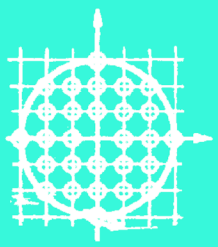

Mathematical Problems: An Anthology

E. B. Dynkin S. A. Molchanov A. L. Rozental A. K. Tolpygo

Workbook 3

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MATHEMATICAL PROBLEMS: AN ANTHOLOGY

BY

E. B. DYNKIN, S. A. MOLCHANOV
A. L. ROZENTAL AND A. K. TOLPYGO

Revised English Edition
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Contents

Preface	vii
Problems	1
Hints and Answers	20
Supplementary Problems	65

Preface

Regardless of their difficulty, the problems in this collection will all yield to the methods of high school mathematics. No attempt has been made to arrange the problems in order of difficulty, and hence there is no need to solve them consecutively. Thus you can dip into the collection at any point and solve the problems in any order.

If you can't solve a problem right away, don't be in a hurry to look up the solution. However, if the problem continues to resist your efforts to solve it, you are then entitled to consult the *Hints and Answers* section. The hints are often brief, but they will be enough to set you on the right track once you have grappled with the problem on your own. The solution should be studied even when you have managed to solve the problem yourself, since it may well turn out that there is an unsuspected gap in your solution. Moreover, the solutions given here often digress and point out interesting side issues.

At the end of the book, you will find a section containing supplementary problems which are not supplied with hints and answers. By all means, try to solve these problems too!

Problems

1. Consider the polygon $ABCDE$ shown in Figure 1. The sides AB, DE and EA are completely visible from the point O, but the side CD is only partly visible. Draw a polygon and a point O inside it such that no side of the polygon is completely visible from O. Draw a polygon and a point O *outside* it such that no side of the polygon is completely visible from O.

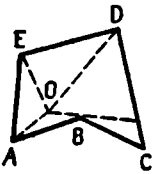

Fig. 1

2. Suppose you want to find out my telephone number (assumed to consist of five digits) by asking me questions which I can only answer by "yes" or "no." What method of interrogation leads to the correct answer after the smallest number of questions?

2a. Solve Problem 2 assuming that I can answer one (but no more) of your questions incorrectly.

3. Two towns A and B are 3 miles apart. It is proposed to build a new school serving 100 students in town A and 50 students in town B. How far from town A should the school be built if the total distance travelled by all 150 students is to be as small as possible?

4. Let M be a point outside a circle with the segment AB as diameter. Using only a ruler, drop a perpendicular from M to the diameter AB (or its continuation).

5. A traveller arrives at an inn. He has no money but only a silver chain consisting of six links. He uses one link to pay for each day spent at the inn, but the innkeeper agrees to accept no more than one severed link. How should the traveller cut up the chain in order to settle accounts with the innkeeper on a daily basis?

6. Let M and N be the midpoints of opposite sides of a quadri-

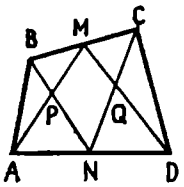

Fig. 2

lateral *ABCD*. Prove that the area of the quadrilateral *MQNP* equals the sum of the areas of the triangles *ABP* and *CDQ* (see Fig. 2).

7. A student belongs to two mathematics clubs located at opposite ends of town, both reached by riding the same subway line but in opposite directions. The student always takes the first train arriving at the station, regardless of the direction in which it is going. At the end of a year, he finds that he has attended the first club twice as often as the second. How is this possible?

Hint. Naturally the student does not always arrive at the subway station at the same time. Rather he arrives sometimes a bit early, sometimes a bit late inside a given interval during which several trains arrive at each side of the platform (see Fig. 3). Assume that the number of "uptown" trains equals the number of "downtown" trains.

Fig. 3

8. Draw a line through the vertex *A* of a convex quadrilateral *ABCD* which divides the quadrilateral into two figures of equal area.

9. (Continuation of Prob. 5). What is the least number of links that have to be cut if the traveller stays 100 days at the inn and has a chain consisting of 100 links? What is the answer in the general case (n days and n links)?

10. The minute and hour hands of a clock coincide exactly at 12 o'clock. At what later time to they first coincide again?

11. One cup contains 5 spoonfuls of tea, another contains 5 spoonfuls of milk. Suppose we pour a spoonful of milk from the second cup into the first, mix the tea and milk carefully and then pour a spoonful of the tea and milk mixture back into the first cup. As a result, each cup contains a mixture of tea and milk, but one contains mostly tea and the other mostly milk. Does the milk contain more tea than the tea contains milk? How is the answer affected by going through the same procedure ten times? What happens if the mixing is not done carefully? What if there is originally more tea in the first cup than milk in the second cup?

12. Suppose a radio tube with seven pins equally spaced along the circumference of a circle is inserted into a socket with seven holes equally spaced along the circumference of a circle. Label both the pins and the holes with the numbers 1, 2, 3, 4, 5, 6, 7 in such a way that at least one pin goes into a hole with the same number regardless of how the tube is inserted into the socket.

13. Prove that the decimal expansion

$$0.12345678910111213141516171819202122\ldots,$$

obtained by writing all the positive integers in order after the decimal point, cannot be periodic.

14. Construct a circle which is equidistant from four given points A, B, C and D. Find all solutions.

15. Show that the product of the digits of any number containing more than two digits is less than the number itself.

16. Suppose there are originally five sheets of paper, some of which are then cut into five smaller pieces. Some of the smaller pieces are themselves cut into five pieces, and so on. After doing

this several times, we stop and count the total number of pieces. Show that there cannot be 1963 pieces.

17. Consider the following game involving two piles of stones: Each of two players in turn takes any number of stones from one of the piles, and the one who takes the last stone wins. What strategy guarantees victory to the player who can either start the game or leave the first move to his opponent?

18. Construct a triangle starting from its three medians.

19. A disk is divided into six sectors, and a checker is placed in each sector. In any given move, two checkers are allowed to move into neighboring sectors, provided one moves clockwise and the other counterclockwise. Can all the checkers end up in one sector?

20. Brown, Jones and Smith are defendants at a trial. One of them has committed a crime. During the trial each makes two statements:

Brown: I am innocent. Jones is innocent.

Jones: Brown is innocent. Smith is guilty.

Smith: I am innocent. Brown is guilty.

It was determined later that one of the two defendants lied twice, one told the truth twice and one lied once and told the truth once. Who committed the crime?

21. A class contains 31 students the sum of whose ages is 434. Are there 20 students in the class the sum of whose ages is less than 280?

22. Delete 100 digits from the number

$$12345678910111213141516 \cdots 5960$$

in such a way as to make the resulting number as small as possible.

22a. Delete 100 digits from the same number in such a way as to make the resulting number as large as possible.

23. Given five points such that no three lie on the same line, prove that four of the points form the vertices of a convex quadrilateral.

24. Trisect the angle 54° using a ruler and compass.

25. Prove that the sum of all fractions with numerator 1 and denominators 2, 3, 4, ..., n cannot be an integer for any value of n.

26. Six delegates meet at an international conference and discover that among any three of them, two speak a common language. Prove that there are three delegates who can all speak the same language.

27. Mrs. Jones has three daughters: Ann, Betty and Clare. One is yachting at Newport, a second is at Bar Harbor, and the third is at Woodstock. One is playing tennis, one is yachting, and one is playing golf. Ann is not at Newport, Clare is not at Bar Harbor, and the daughter who plays golf is not at Woodstock. If the yachting enthusiast is not Clare, who is playing golf and where?

28. Two lines are drawn through the points of intersection A and B of two circles. Let C and D be the points in which the two lines intersect the first circle, and let E and F be the points in which they intersect the second circle. Prove that the lines CD and EF are parallel.

29. Can the number 25 be written as a sum of ten terms each equal to 1, 3 or 5?

30. There are six points in the plane no three of which lie on the same line. Prove that three of the points form a triangle containing an angle of 120° or more.

31. Find all triples of integers x, y and z satisfying the equality

$$x^2 + y^2 + z^2 = 2xyz.$$

32. Construct the quadrilateral $ABCD$ starting from the four sides and the angle between two sides AB and CD.

33. A city has several (more than one) bus lines such that

1) There are exactly three stops on each line;
2) Any two lines have exactly one stop in common;
3) There is just one line connecting any given stop to any other stop.

How many bus lines are there?

34. Two identical regular octagons are cut out of cardboard. The vertices of one are labelled from 1 to 8 consecutively in the counterclockwise direction. Can the vertices of the second octagon be labelled from 1 to 8 in such a way that regardless of how the second octagon is superimposed on the first, one vertex always lands on another vertex with the same number?

35. Given an isosceles triangle ABC, suppose the angles BAC and BCA equal 80°. From the vertices A and C draw two lines intersecting the opposite sides in the points D and E, respectively. The angle CAD equals 60°, while the angle ACE equals 50°. Find the angle ADE.

36. A tent is erected on four stakes which come together at the top of the tent. Prove that it is always possible to drive a nail into each stake in such a way that the nails are at the vertices of a parallelogram.

37. Prove that if the bisectors of two angles of a triangle are equal, then the bisected angles are equal.

38. Prove that there is no straight line intersecting all the sides of a polygon with 1001 sides.

39. The sum of the digits of a number does not change when the number is multiplied by 5. Prove that the number is divisible by 9.

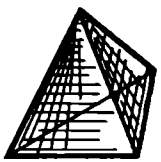

Fig.4

40. A dairy official who is an amateur mathematician offers a prize to anybody who can design a milk carton shaped like a triangular pyramid (see Fig.4) such that at least one obtuse angle is adjacent to every common side of two triangles. Will the prize ever be won?

41. Prove that given any hundred whole numbers we can always find one or more numbers whose sum ends in two zeros.

42. In how many zeros does the product of the hundred integers from 1 to 100 end?

43. How many different ways are there of writing 1,000,000 as a product of three factors other than 1? (Products differing only in the order of factors are not regarded as distinct.)

44. Find the point in a convex quadrilateral for which the sum of the distances to the vertices is a minimum.

45. A snail crawls away from the point A, turning 90° in some direction every 15 minutes. Prove that the snail can only return to the point A after an integral numbers of hours. (The snail's speed is assumed to be constant.)

46. Prove that there are no whole numbers x and y satisfying the equation

$$15x^2 - 7y^2 = 9.$$

47. Three rugs of arbitrary shape, each with an area of 3 square yards, are spread on the floor of a room whose area is 6 square yards. Prove that two rugs overlap and share an area of at least one square yard.

How is the answer affected if four rugs, each with an area of 2 square yards, are spread on the same floor?

48. The plane is divided into square cells like a sheet of graph paper, and a whole number is written in each cell. Suppose each number equals the arithmetic mean of the numbers appearing in the neighboring cells (above, below, to the right and to the left of the given cell).* Prove that the numbers are all equal.

49. A certain whole number contains 300 ones, but all the other digits are zeros. Can the number be a complete square?

50. It is not hard to cover the 64 squares of a chessboard with 32 dominoes in such a way that every domino covers two squares (provided, of course, that the dimensions of the dominoes match those of the squares). Can 62 squares of a chessboard be covered with 31 dominoes in such a way that the two squares in the upper left-hand and lower right-hand corners (or in the lower left-hand and upper right-hand corners) are left uncovered?

---

* By the *arithmetic mean* of the four numbers a, b, c and d is meant the quantity

$$\frac{a+b+c+d}{4}.$$

51. Find the greatest common divisor of the numbers $a = 11{,}111{,}111$ and $b = 1{,}111, \ldots, 111$ (the digit 1 is repeated 100 times).

52. Thirty points are chosen inside a convex 100-gon (a polygon with 100 sides) such that no three points lie on the same line. The 100-gon is cut into triangles in such a way that the set of vertices of all the triangles consists of the thirty chosen points and the 100 vertices of the original polygon. How many triangles are there?

53. Given a segment AB and a line intersecting AB, construct the triangle ABC with the line as the bisector of the angle ACB.

54. (A statistical experiment.) Open any book and consider any passage consisting of 1000 letters (ignore spaces). Count the number of times each letter occurs in the passage. Then arrange the letters in order of decreasing frequency of occurrence. Calculate the average number of letters in a word.

55. Decipher the following coded message:

"E mevv fuz pu ku nyj yk zu kya zqyz zu xufkzjoxz y qekzuja un zquopqz mezquoz gjunuofb kzoba un zqi wyzqiwyzexyv ebiyk un koxxikkeli iguxqk ek vesi uwezzefp Qywviz njuw zqi gvya mqexq ek fywib ynzij qew. Zqyz muovb di xvyewefp zuu woxq. Doz ez ek xijzyefva yfyvupuok zu xozzefp uoz zqi gyjz un Ugqivey. Zqek kewevi ek kefpovyjva icyxz. Nuj Ugqivey ek hoezi ikkifzeyv zu zqi gvya, kqi ek lija xqyjwefp yfb y vezzvi wyb." Mqeziqiyb

Hint. The message is encoded as follows: The vowels a, e, i, o, u and the letter y are grouped in pairs somehow, and so are the remaining consonants. Each letter in the message is replaced by the other letter belonging to the same pair.

56. An urn contains 100 balls of different colors, i.e., 28 red, 20 green, 12 yellow, 20 blue, 10 white and 10 black. What is the smallest number of balls that must be drawn from the box without looking if 15 balls are to be of the same color?

57. Construct a square given one point on each of its sides.

58. Can the sum of the squares of two consecutive positive

integers equal the sum of the fourth powers of two other consecutive positive integers? In other words, do there exist two positive integers m and n such that

$$m^2 + (m + 1)^2 = n^4 + (n + 1)^4?$$

59. Find all pairs of positive integers m and k such that the sum of the integers from m to $m + k$ inclusive equals 1000.

60. Suppose the median, bisector and altitude drawn from the same vertex of a triangle divide the vertex angle into four equal parts. Find the angles of the triangle.

61. Prove that if $a + b + c = 1$, then

$$a^2 + b^2 + c^2 \geqslant \tfrac{1}{3}.$$

62. Prove that in any set of 101 numbers between 1 and 200, there are always two numbers one of which divides the other.

63. Given any positive integer N, prove that there is a number divisible by N all n digits of which are zeros or ones. (What is the smallest value of n that will do?)

64. Prove that if the median of a triangle coincides with a bisector of one of its angles, then the triangle is isosceles.

65. Given a parallelogram $ABCD$, draw the segments joining the vertex A to the midpoints E and F of the opposite sides BC and CD. Let M and N be the points in which these segments intersect the diagonal BD. Prove that M and N divide BD into three equal parts.

66. Among numbers all of whose digits are ones, find the smallest number N divisible by

$$\underbrace{33\cdots33}_{\text{100 times}}.$$

67. Consider a square table consisting of 16 cells, each containing a plus or a minus. Suppose we change all the signs in a given row or column to their opposites, doing this several times until the number of minuses is a minimum. Let the

smallest number of minuses to which a given table can be reduced in this way be called its *characteristic*. What values can a characteristic have? (Find all possible values.)

68. Into how many parts is a convex 20-gon (polygon with 20 sides) divided by drawing all its diagonals, if no three diagonals intersect in a single point?

69. An arbitrary line L is drawn through the center of a square one unit on a side. Calculate the sum of the squares of the distances from L to the four vertices of the square.

70. Write the polynomial $x^8 + x^4 + 1$ as a product of four factors.

71. One hundred students belong to a mathematics club. Suppose that among any four students there is at least one who knows the other three. Prove that there is a student who knows all 99 other students. What is the minimum number of such students?

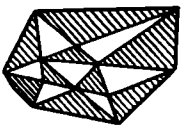

Fig. 5

72. Figure 5 shows a hexagon partitioned into black and white triangles such that

1) Two triangles either have a common side (then they are of different colors) or a common vertex, or else they have no points in common;
2) Every side of the hexagon is also a side of one of the black triangles.

Prove that a decagon cannot be partitioned in this way.

73. A regular decagon is inscribed in a circle of radius 1. Find the sum of the squares of the distances from any point of the circle to all the vertices of the decagon.

74. Suppose the result of multiplying a four-digit number by the same number written in reverse order is an eight-digit number whose last three digits are zeros. Find all such numbers.

75. Construct a triangle, given the base, vertex angle and the median drawn to a lateral side.

76. Given that six disks have a point in common, prove that at least one disk contains the center of another disk.

77. Consider all possible seven-digit numbers made up of the

numbers 1, 2, 3, 4, 5, 6, 7 written in any order. Prove that none of these numbers is divisible by any of the others.

78. The vertex A of an acute triangle ABC is joined by a line segment to the center O of the circumscribed circle. Then the altitude AH is drawn from the vertex A. Prove that $\angle BAH = \angle OAC$.

79. Prove that

$$x^{12} - x^9 + x^4 - x + 1 > 0$$

for arbitrary x.

80. Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{199} - \frac{1}{200} = \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{200}.$$

81. The faces of a cube can all be painted white or all black, or else some can be painted white and others black. How many different ways of painting the cubes are there? (Two cubes are regarded as painted differently if they cannot be confused regardless of orientation.)

82. A rectangular board made up of 4×100 squares is covered by dominoes in such a way that

1) Each domino covers precisely two squares;
2) No two dominoes overlap;
3) No square is left uncovered.

Prove that the board can be cut along one of its longitudinal or transverse rulings without moving or cutting any of the dominoes.

83. Given an equilateral triangle ABC, suppose another equilateral triangle CDE is constructed on the extension CE of the side AC. Let M be the midpoint of the segment AD, and let P be the midpoint of the segment BE (see Fig. 6). Prove that the triangle CMP is equilateral.

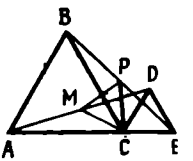

Fig. 6

84. Twenty problems are assigned to a class of twenty students. It turns out that every student has solved two problems,

while every problem has been solved by two students. Prove that the problem discussion can be organized in such a way that every student discusses one of the problems assigned him and every problem is discussed.

85. Suppose a circle can be inscribed in a certain decagon. Construct the decagon, given all its angles and one of its sides.

86. Prove that the last digits of the numbers

$$\frac{1 \cdot 2}{2}, \frac{2 \cdot 3}{2}, \frac{3 \cdot 4}{2}, \ldots, \frac{n(n+1)}{2}$$

repeat periodically.

87. The numbers 1 through 64 are written consecutively in the 64 squares of a chessboard (i.e., the numbers 1 through 8 are written in order from left to right in the top row, the numbers 9 through 16 in the next row, etc.). Eight queens are placed on the board so that no two threaten each other. Find all possible values of the sum of the numbers written in the eight squares occupied by the queens.

88. How many times does the number 2 appear when the product

$$(n+1)(n+2)\cdots(2n-1)2n$$

is expanded in prime factors?

89. Given an isosceles triangle ABC, drop the perpendicular HE from the midpoint H of the base BC to the lateral side AC. Let O be the midpoint of the segment HE. Prove that the lines AO and BE are perpendicular.

90. Let A be a point inside a circle but distinct from its center. Can one always direct a ray of light from A so that it returns to A after being reflected from the circle at several (more than two) points?

Hint. Upon reflection of the ray from the circle, the angle of reflection equals the angle of incidence, i.e., the reflected and incident rays form equal angles with the radius drawn to the point of incidence.

91. Suppose a closed polygonal curve consists of 203 segments, no two of which lie on the same line. What is the largest possible number of points in which the curve intersects itself?

92. Given five circles, every four of which have a point in common, prove that all five circles have a point in common.

93. Let A and B be points lying on two straight lines intersecting in a point O. Suppose two pedestrians leave A and B and move towards O with the same velocity. Find the least distance between the moving pedestrians. How long does it take the pedestrians to be the least distance apart if $\angle AOB = 90°$, $AO = a$, $BO = b$?

94. Suppose a quadrilateral is divided into triangles of equal area by each of its diagonals. Prove that the quadrilateral is a parallelogram.

95. The numbers from 1 to 18 are placed at the 18 points shown in Figure 7. Prove that there is a segment such that the numbers at the end points of the segment differ by more than 3.

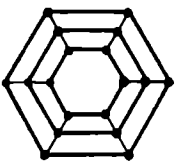

Fig. 7

96. Prove that if $m \neq n$, then the numbers $2^{2^m} + 1$ and $2^{2^n} + 1$ have no common factors.

97. Draw a line dividing a given trapezoid into two similar quadrilaterals.

98. Given arbitrary positive numbers a, b and c, prove that the inequalities

$$a(1 - b) > \tfrac{1}{4},\ b(1 - c) > \tfrac{1}{4},\ c(1 - a) > \tfrac{1}{4}$$

cannot all hold simultaneously.

99. Prove that $(a + b)^4 \leqslant 8a^4 + 8b^4$.

100. A rod AB of length a is hinged at the point B to a rod BC of length b. An equilateral triangle ACE is constructed with the segment AC as its base. For what configuration of the rods is the distance BE largest?

101. Prove that among all the rectangles inscribed in a semicircle, the rectangle with one side four times larger than the other has the largest perimeter.

102. Let ABC be any triangle which is not equilateral. Prove that ABC can be covered entirely by two smaller triangles similar to ABC.

103. A point inside an isosceles trapezoid is joined to all the vertices. Prove that the four resulting segments can be used to form a quadrilateral inscribed in the trapezoid (with one vertex of the quadrilateral on each side of the trapezoid).

104. A small disk and a large disk are cut out of cardboard. Each disk is divided into 200 equal sectors by drawing appropriate radii, with 100 sectors on each disk colored black and the other 100 sectors left white. The disks are then superimposed with their centers coinciding. Prove that the inner disk can be rotated in such a way that at least 100 of its sectors lie on top of sectors of the outer disk which have the same color.

105. Prove that if three numbers x, y and z satisfy the system of equations

$$x + y + z = a,$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a},$$

then at least one of the numbers equals a.

106. All the positive integers are written in order, starting with 1. Find the digit appearing in the 206,788th place.

107. Given a square table consisting of 25×25 cells, suppose one of the first 25 positive integers 1, 2, ..., 25 is written in each of the 625 cells in such a way that

1) The same number is written in any two cells symmetric with respect to the main diagonal;
2) Two identical numbers cannot appear in the same row or in the same column.

Prove that the numbers in the main diagonal are all distinct.

108. Given three points A, B and C in the plane and three angles H, K and M, suppose each angle is less than 180° while the sum of all the angles equals 360°. Using a ruler and protractor, construct a point O such that $\angle AOB = H$,

$\angle BOC = K$, $\angle COA = M$. (A protractor can be used to transfer any given angle from its original position to another in which one side of the angle has an arbitrarily preassigned position.)

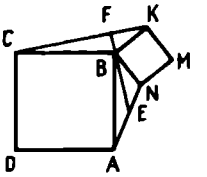

Fig.8

109. Let *ABCD* and *BKMN* be two squares. Prove that extending the median *BF* of the triangle *ABN* past the vertex *B* gives the altitude of the triangle *KBC* (see Fig.8).

110. Prove that a chessboard 10 squares on a side cannot be covered (without overlapping) by "tiles" like the one shown in Figure 9. It is assumed that tiles are laid in such a way that each of the four squares in a tile covers a square of the board (the squares in the tiles are of the same size as those of the board).

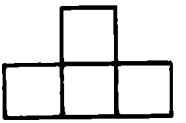
Fig.9

111. Given $n > 3$ points in the plane, no three of which are collinear, is it always possible to construct a circle passing through at least 3 of the n points such that none of the n points lies inside the circle?

112. Prove that a knight cannot be moved on a $4 \times n$ chessboard in such a way as to return to the original square after landing on every square just once.

113. Given n points in the plane every four of which lie at the vertices of a convex quadrilateral, prove that the n points lie at the vertices of a convex n-gon.

114. Prove that there are infinitely many positive integers which cannot be written as sums of cubes of three positive integers.

115. Prove that the number $10^n + 18n - 1$ is divisible by 27 for any positive integer n.

116. Prove that the perimeter of a triangle exceeds the sum of the lengths of its three medians.

117. In the number

$$30_0_03$$

choose the two missing digits in such a way that the result is divisible by 13.

118. Find the result of dividing $a + a^2 + \cdots + a^{100}$ by $a^{-1} + a^{-2} + \cdots + a^{-100}$.

119. Find every two-digit number whose square equals the cube of the sum of its digits.

120. Solve the system of equations

$$x^3 - 3x^2y - xy + 2x - 2 = 0,$$

$$x^3y - 3x^2y^2 - xy^2 + 2xy + x - 2y - 2 = 0.$$

121. Prove that if a prime number is divided by 30, then the remainder is itself prime.

122. By the *Fibonacci sequence* is meant the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

where every term beginning with the third is the sum of the preceding two terms. Find the greatest common divisor of the 770th and 1000th terms of the Fibonacci sequence.

123. Describe all numbers with the following property: The number obtained by writing the digits in reverse order divides the original number (with a quotient distinct from 1).

Comment. This problem is due to G.E. Shilov. It is assumed that the original number has more than one digit and does not end in a zero, so that the number obtained by reversing its digits begins with a significant figure.

124. Suppose that a_1 students taking a certain mathematics course fail at least one quiz during the term, a_2 students fail at least 2 quizzes, and so on, with a_k students failing at least k quizzes. How many quizzes are marked "F"? (It is assumed that no student fails more than k quizzes.)

125. Find the smallest positive integer N such that $\frac{1}{2}N$ is a perfect square, $\frac{1}{3}N$ is a perfect cube and $\frac{1}{5}N$ is a perfect fifth power.

126. Let $n?$ denote the product of all prime numbers not exceeding n ($n \geqslant 2$). Find all values of n such that $n? \leqslant n$.

127. Given a convex quadrilateral, find a point inside the quadrilateral such that the segments joining the point to the

midpoints of the sides divide the quadrilateral into four parts of equal size.

128. The vertex P of an equilateral triangle PKM is fixed while the vertex K describes a square Q. What geometric figure is described by the third vertex?

129. Given two segments AB and CD, find a point O such that the angles AOB and COD are equal while the triangles AOB and COD are similar.

130. Numbers are placed in the cells of a square table of size $n \times n$. The sum of the numbers appearing in any "cross" is no less than a. (By a "cross" is meant a figure consisting of one complete row and one complete column.) What is the smallest value of the sum of all the numbers in the table?

131. A million lines are drawn in the plane such that

1) No two lines are parallel;
2) At least one other line goes through the point of intersection of any two lines.

Prove that all the lines go through one point.

132. Suppose the sun is at its zenith. How should a match box be held in order to cast the shadow of largest area on a horizontal table? How about a triangular pyramid?

133. Six numbers, among them 0 and 1, are written on the six faces of a cube. Each of the six numbers is replaced by the arithmetic mean of the four numbers on the four adjacent faces. The same operation is carried out again for the new numbers, and so on, for a total of 25 times in all. Suppose it turns out that, as a result, every face is labelled with the same number as before. Prove that there must be a mistake in the calculations.

134. The eight vertices of a cube are labelled with eight numbers, among them 0 and 1. Each of the eight numbers is then replaced by the arithmetic mean of the three numbers at the three adjacent vertices. After the same operation has been carried out a total of 10 times, it turns out that each vertex is labelled with the same number as originally. Find the original eight numbers.

135. Find the point of a triangle for which the sum of the distances to the three sides is a) smallest; b) largest.

136. For which three distinct positive integers k, l and m is the sum

$$\frac{1}{k}+\frac{1}{l}+\frac{1}{m}$$

less than $\frac{1}{2}$ but as close as possible to $\frac{1}{2}$? For which four distinct positive integers, k, l, m and n is the sum

$$\frac{1}{k}+\frac{1}{l}+\frac{1}{m}+\frac{1}{n}$$

less than 1 but as close as possible to 1?

137. What is the largest number of times that a ray can be reflected from the sides of an angle of 1°? (At each reflection, the angle of incidence equals the angle of reflection.)

138. A ray of light is reflected repeatedly from the sides of an equilateral triangle. Suppose that during a certain time the ray is reflected 4 times from the side AB, x times from the side AC and y times from the side BC. What values can x and y have? (Discuss all possibilities.)

139. Given a chessboard made up of $100 = 10 \times 10$ squares, consider a piece called a "lion" which can move one square to the right, or one square downward, or one square diagonally upward to the left. Can a lion move over the whole board, returning to its original position after occupying each square just once?

140. By "double chess" is meant the game differing from ordinary chess in that each of the players makes two consecutive moves. Prove that the first player (White) can always manage to win or at least draw.

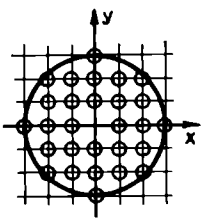

Fig. 10

141. Consider a circular park of radius s, regularly planted with trees as shown in Figure 10, i.e., in such a way that the centers of the trees are at points whose coordinates are integers (however, there is no tree at the origin of coordinates). To what

diameter should the trunks of the trees be allowed to grow in order to completely block the view from the center? Prove that the view is blocked if the radius of every tree exceeds $1/s$ but not if the radius is less than $1/\sqrt{s^2 + 1}$.

142. There are a number of countries occupying a spherical planet, each country consisting either of a single connected part or of two disconnected parts. It is required to make a map of the planet such that

1) Each country is of one color (if the country has two parts, then both parts are of the same color);
2) Every country sharing a common boundary with a given country is of a different color.

Prove that 12 colors always suffice but 11 colors may not be enough.

143. There are a number of countries occupying two spherical planets. Some countries have territory on only one planet, while others have part of their territory on one planet and part on the other planet. (The territory of each country on each planet is connected.) Prove that 12 colors always suffice to make a map of the two planets satisfying the two conditions given in the preceding problem. However, prove that 7 colors may not be enough.

Hints and Answers

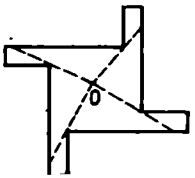

Fig. 11

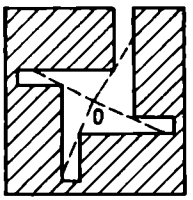

Fig. 12

1. One possible solution is shown in Figure 11. A slight modification of this figure leads to a solution of the second part of the problem (see Fig.12).

2. Each successive question should reduce the number of remaining possibilities by two. Clearly, with such a method of interrogation, n questions suffice to uniquely determine one of 2^n possibilities. Since by hypothesis there are a total of $10^5 < 2^{17}$ telephone numbers, 17 questions will do. There is considerable freedom of choice in the questions themselves. For example, you might ask first "Is your number larger than 50,000?" If I answer "yes", your second question might be "Is it larger than 75000", and so on. But the simplest thing is to base the questions on the binary number system, in which there are only two digits 0 and 1 and every number less than 100,000 involves no more than 17 such digits. You might then ask:

1) "Is the last digit of your telephone number 1 when written in binary form?"
2) "Is the next to the last digit 1 when written in binary form?" and so on.

It is easy to see that 16 questions are not enough. In fact, there are 2^{16} distinct combinations each consisting of 16 occurrences of the words "yes" and "no." But $100{,}000 > 2^{16}$, and hence two distinct telephone numbers lead to the same series of questions and we have no way of telling which is the actual telephone number.

The principle of "equipartition" used here is useful in many problems, for example in the familiar game of "twenty questions," where one person thinks of the name of a famous man

and the other tries to guess this name by asking questions which can only be answered by "yes" or "no." Then one must try to ask questions to which either a "yes" or a "no" answer seems equally likely. For example, it is bad to start with the question "Is he a biologist?" It is far better to ask "Is he a scientist?"

2a. 22 questions are enough to solve the problem. One method of interrogation is the following: First you use 15 questions to determine 15 of the 17 digits in the binary representation of the telephone number. Then you ask "Was one of your 15 answers a lie?" If I answer "yes", then either one of the first 15 answers is a lie, or else the last answer is a lie. By the method of "equipartition", you can find out which of the 16 answers is a lie by using only 4 questions, and two more questions are then enough to determine the last two digits. However, if my answer to the sixteenth question is "no", then I actually did not lie and you have correctly ascertained the first 15 digits (otherwise the sixteenth answer and one of the first 15 answers would both be lies, contrary to the assumption that I am allowed only one lie). Five questions are then enough to determine the remaining two digits. For example, you might ask me about each digit twice and then ask me a third time if two of my answers contradict each other.

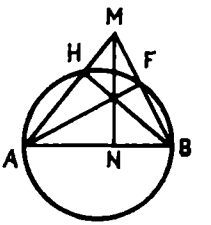

Fig. 13

3. The school should be built in town *A*.

4. The solution is shown in Figure 13.

5. If the third link is cut, the chain falls into three pieces (see Fig.14). Using them, the traveller can settle his accounts

Fig. 14

day by day, since the innkeeper can then give change by using links received earlier.

6. First prove that the altitude of the triangle *AMD* drawn to *AB* equals the sum of the altitudes of the triangles *ABN* and

NCD. Deduce from this that the area of the triangle *AMD* equals the sum of the areas of the triangles *ABN* and *NCD*.

7. Suppose, for example, that trains go in each direction at three minute intervals, and suppose the schedule is such that there is a one minute wait for an uptown train if you just miss a downtown train but a two minute wait for a downtown train if you just miss an uptown train (see Fig.15, where heavy dots

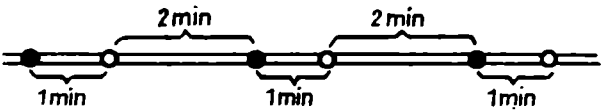

Fig.15

indicate passage of downtown trains and light dots indicate passage of uptown trains). Then it is twice as probable that you will arrive at the station during a two minute interval as during a one minute interval, and hence twice as probable that you will take a downtown train instead of an uptown train.

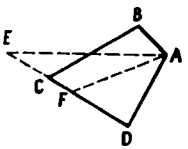

Fig.16

8. Suppose the area of the triangle *ABC* is less than that of the triangle *ACD*. Let *ACE* be the triangle whose area equals that of *ABC* and whose base *EC* lies on the continuation of *DC*. Then draw the median *AF* of the triangle *EAD* (see Fig.16).

9. First note that if the traveller has a chain consisting of $n = (k + 1)\,2^{k+1} - 1$ links, he can cut k links in such a way as to obtain pieces consisting of $(k + 1), 2(k + 1), 2^2(k+1), \ldots, 2^k(k + 1)$ links. Then, using these pieces and the k severed links, the traveller can settle accounts with the innkeeper on a daily basis for n days. If the number of links in the chain is not of the form $n = (k + 1)\,2^{k+1} - 1$, we have to find the smallest integer k such that $n < (k + 1)\,2^{k+1} - 1$. In particular, only 4 links need be cut if $n = 100$.

10. The hands first coincide again at $5\frac{5}{11}$ minutes past one.

11. The problem does not require detailed explanation. Clearly the amount of milk entering one cup equals the amount of tea leaving it. Therefore the amount of tea in the milk always

equals the amount of milk in the tea, regardless of the extent of mixing or the number of transfers from one cup to the other.

12. Label the pins with the numbers 1, 2, 3, 4, 5, 6, 7 arranged in clockwise order, and label the holes with the same numbers arranged in counterclockwise order. There are other ways of labelling the pins and the holes.

13. Suppose there is a period consisting of n digits. Consider the sequence obtained by writing all the positive integers in order. Arbitrarily far from the beginning of this sequence there are numbers containing n zeros in a row. Therefore the period must consist of zeros only. A similar argument shows that the period consists of ones only. Contradiction!

14. Show that the center of the circle can only be the point of intersection of the perpendiculars to the midpoints of the diagonals AC and BD of the quadrilateral $ABCD$ or the points of intersection of the perpendiculars to the midpoints of any two sides of $ABCD$.

15. Write the number in the form

$$N = a_0 10^n + a_1 10^{n-1} + \cdots + a_n,$$

and note that

$$a_0 a_1 \cdots a_n \leqslant a_0 10^9.$$

16. Show that the number of pieces at the end of each stage of subdivision is of the form $4k + 5$.

17. Each time the player should take enough stones from the pile containing more stones to make the number of stones the same in both piles. If both piles contain the same number of stones in the beginning, the player should allow his oppenent to move first.

18. Suppose the problem has already been solved, and let ABC be the desired triangle. Extend the median BM by the segment $MF = OM$ (see Fig.17). Then use the proportionality between the sides of the triangle AOF and the medians of the triangle ABC.

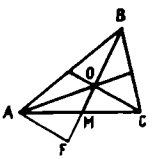

Fig. 17

19. Number the sectors in order and consider the sum $k_1 + 2k_2 + \cdots + 6k_6$, where k_1 is the number of checkers in the

first sector, k_2 the number in the second sector, and so on. Note that after each move the sum either does not change at all or else changes by 6. The sum is originally equal to 21.

20. Brown committed the crime. Jones lied twice and Smith told the truth twice.

21. Consider the sum of the ages of the 20 oldest students.

22. Note that 11 digits remain after the deletion. The digits can be deleted un such a way as to leave the number 00000123450. Prove that no smaller number can be obtained.

22a. Delete the digits in such a way as to leave the number 999999785960.

23. Consider any four points and show that they either form a convex quadrilateral or else one of them lies inside the triangle formed by the other three. In the latter case, the fifth point can fall in one of the nine regions shown in Figure 18. Show that a

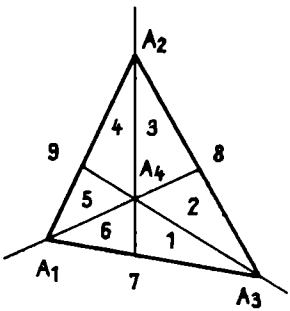

Fig. 18

convex quadrilateral can be constructed for any position of this point.

24. Consider the complementary angle.

25. Consider the smallest number of the form $\frac{1}{2^m}$ contained in the sum $\frac{1}{2} + \cdots + \frac{1}{n}$ and show that when reducing $\frac{1}{2} + \cdots + \frac{1}{n}$ to its least common denominator the term $\frac{1}{2^m}$

contributes an odd number to the numerator while all the other terms contribute even numbers.

26. Suppose delegate A can talk with three other delegates, say B, C and D. Then among the latter at least two, say B and C, can also talk to each other. Then A, B and C all speak a common language. On the other hand, if A can understand no more than two other delegates, then there are three delegates, say E, F and G, none of whom A understands. But then E, F and G all speak a common language (why?).

27. Ann is playing golf at Bar Harbor.

28. Consider the angles of the quadrilaterals $ABCD$ and $ABEF$.

29. The number of terms must be odd.

30. If the points lie at the vertices of a convex hexagon, the problem is easily solved by noting that the sum of the angles of a convex hexagon equals 720° and hence at least one angle is 120° or more. We must still consider the case where 3, 4 or 5 of the points lie at the vertices of a convex polygon and the other vertices lie inside the polygon. The diagonals drawn from some vertex of the polygon divide the polygon into triangles, and one of these triangles contains one of the given points. By joining the point to the vertices of the triangle we obtain three angles whose sum equals 360°. Therefore at least one of the angles is 120° or more.

31. If one of the numbers x, y, z equals zero, then the others obviously equal zero. Suppose x, y and z are nonzero. Then $x = 2^m x_1$, $y = 2^n y_1$, $z = 2^p z_1$, where the numbers x_1, y_1 and z_1 are odd. Suppose, for example, that $m < n \leqslant p$. Then both sides of the equality can be divided by $(2^m)^2$ and the result can be written in the form

$$u^2 + v^2 + w^2 = 2^{m+1} uvw \quad (m \geqslant 0),$$

where u is odd and v and w are even. But the right-hand side is divisible by 4 while the left-hand side is not. Contradiction! The case $m \leqslant n \leqslant p$ also leads to a contradiction.

32. From the point C draw a segment CF equal to and parallel to AB. Now consider the triangle CFD.

33. First we prove two lemmas:

Lemma 1. Given any stop K, there is a bus line which does not go through K.

Proof. If there is only one line going through a given stop K, then none of the other lines goes through K. Suppose two lines, say KL and KM (see Fig. 19a), go through K. (The letters L

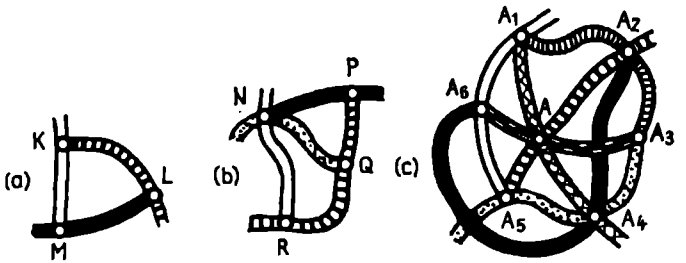

Fig. 19

and M denote distinct stops different from K.) Because of condition 3, there is a line LM going through L and M. This line does not go through K. In fact, the line LM is distinct from KL or KM, and according to condition 2, two distinct lines have only one stop in common. This proves Lemma 1.

Lemma 2. Three and only three bus lines go through every stop.

Proof. One can go from stop N (see Fig. 19b) to stop P of the line PQR by taking some line PN (condition 3) which does not go through Q and R (condition 2). A similar argument shows that there must be lines NQ and NR distinct from NP and each other. Every line going through N has a stop in common with line PQR and hence coincides with one of the three lines NP, NQ and NR. This proves Lemma 2.

Now consider the three bus lines going through a given stop N. Each of these lines has two more stops, and according to condition 3, there is just one line connecting any given stop

to any other stop. Hence there are 7 stops in all. A total of $7 \cdot 3 = 21$ lines go through the 7 stops, but each line is counted three times, since it goes through 3 stops. In other words, there are 7 lines in all, connected in the way shown in Figure 19c.

34. Assuming that the answer is yes, superimpose the octagons in such a way that the two numbers 1 coincide. Suppose that a_i ($a_i = 1, 2, \ldots, 8$) is the number on the lower octagon opposite the number i on the upper octagon. Then the number i on the upper octagon can be made to coincide with the number i on the lower octagon by rotating the upper octagon through the angle $b_i \cdot 45°$ in the counterclockwise direction, where

$$b_i = \begin{cases} i - a_i & \text{if } i > a_i, \\ i - a_i + 8 & \text{if } i \leqslant a_i. \end{cases}$$

The b_i must take all the values 1, 2, ..., 8 (why?). Adding the b_i, we obtain

$$b_1 + b_2 + \cdots + b_8 = (1 + 2 + \cdots + 8) - (a_1 + a_2 + \cdots + a_8) + 8k,$$

where k is some positive integer. But

$$a_1 + a_2 + \cdots + a_8 = b_1 + b_2 + \cdots + b_8 = 1 + 2 + \cdots + 8 = 36,$$

which is impossible since 36 is not divisible by 8.

35. From the vertex C draw a line making an angle of 60° with AC and intersecting AB in the point F and AD in the point O (see Fig.20). Prove that the triangles EFD and EOD are congruent. It follows at once that the angle ADE equals 30°.

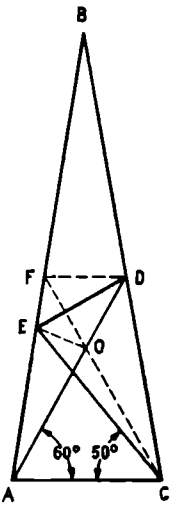

Fig.20

36. Let OE be the line in which the planes AOC and DOB intersect. Through any point on the line OE draw the lines KL (in the plane AOC) and MN (in the plane DOB) such that OE is the median of the two resulting triangles (see Fig.21).

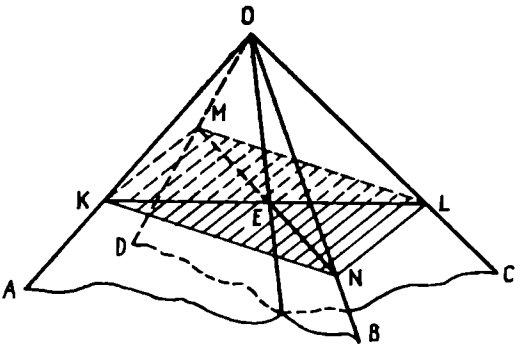

Fig. 21

Alternatively, draw the plane parallel to the lines of intersection of opposite faces of the tent and consider the intersection of this plane with the tent.

37. First show that two triangles are congruent if they have equal bases, vertex angles and bisectors of the vertex angles. To prove this lemma, superimpose the bases of the triangles

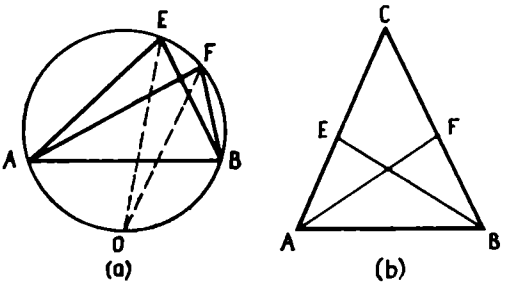

Fig. 22

and extend the bisectors until they intersect the circumscribed circle (see Fig. 22a). Then use the lemma to show that the triangles *AFC* and *BEC* are congruent if the bisectors *AF* and *BE* are equal (see Fig. 22b).

38. Note that if such a line existed, then there would be the same number of vertices of the polygon on both sides of the line.

39. Prove that a number and the sum of its digits have the same remainder when divided by 9. Deduce from this that the difference between two numbers whose digits have the same sum is divisible by 9.

40. Show that there can be no obtuse angle adjacent to the largest edge of the carton.

41. Denote the given numbers by $n_1, n_2, \ldots, n_{100}$ and consider the sums $S_1 = n_1, S_2 = n_1 + n_2, S_3 = n_1 + n_2 + n_3, \ldots, S_{100} = n_1 + n_2 + \cdots + n_{100}$. Suppose none of these sums is divisible by 100. Then dividing each of them by 100 can only leave one of the remainders 1, 2, 3, ..., 99 and hence two of the sums leave the same remainder. Suppose, for example, that these sums are S_{21} and S_{87}. Then

$$n_{22} + n_{23} + n_{24} + \cdots + n_{86} + n_{87} = S_{87} - S_{21}$$

is divisible by 100.

42. The number of zeros at the end of a number shows how many times 10 is a factor of the number. The number 10 equals the product $2 \cdot 5$, but the factor 2 occurs more often than the factor 5 in the product $1 \cdot 2 \cdot 3 \cdots 100$. Therefore $1 \cdot 2 \cdot 3 \cdots 100$ is divisible by the same power of 10 (i.e., ends in the same number of zeros) as the number of factors of 5 in the product $1 \cdot 2 \cdot 3 \cdots 100$. But there are 20 numbers from 1 to 100 which are multiples of 5, and four of these (25, 50, 75 and 100) are also multiples of $25 = 5^2$, i.e., contain two factors of 5. Therefore $1 \cdot 2 \cdot 3 \cdots 100$ contains 5 as a factor 24 times, i.e., $1 \cdot 2 \cdot 3 \cdots 100$ ends in 24 zeros.

43. 162 ways.

44. The point of intersection of the diagonals.

45. Show that in order to return to A, the snail must crawl for the same amount of time in each of the four directions.

46. If $7y^2 = 15x^2 - 9$, then y^2 and hence y must be divisible by 3. Let $y = 3u$. Then

$$5x^2 = 3 + 21u^2,$$

which implies $x = 3v$. But then

$$7u^2 + 1 = 15v^2.$$

Show that a number of the form $7u^2 + 1$ is never divisible by 3.

47. Suppose the area shared by the first and second rugs is S_1, that shared by the first and third rugs is S_2, and that shared by the second and third rugs is S_3, while that shared by all the rugs is S_4. Show that

$$6 = 9 - (S_1 + S_2 + S_3) + S_4.$$

48. Let a be the smallest number in the array, and let p, q, r, s be the numbers in the four adjacent cells. Since $4a = p + q + r + s$, we have

$$(p - a) + (q - a) + (r - a) + (s - a) = 0.$$

But the numbers in parentheses are nonnegative, and hence

$$a = p = q = r = s.$$

It follows that all the numbers in the array are equal.

49. A number containing 300 ones and any number of zeros is divisible by 3 but not by 9. Therefore it cannot be a perfect square.

50. No, because each domino covers one white and one black square, whereas two squares of the same color are excluded from the board.

51. Clearly $b = aq + 1111$, i.e., $b - aq = 1111$. Therefore any common divisor of the numbers a and b, including the greatest common divisor, is a divisor of 1111. But the number 1111 is itself a divisor of a and b, and hence must be their greatest common divisor.

More generally, we might ask for the greatest common divisor d of the numbers

$$a = \underbrace{11\cdots11}_{m \text{ times}}, b = \underbrace{11\cdots11}_{n \text{ times}} \quad (m < n).$$

It turns out that*

$$d = \underbrace{11 \cdots 11}_{p \text{ times}},$$

where p is the greatest common divisor of m and n.

52. Calculate the sum of the interior angles of all the triangles: The sum of the angles of the triangles with vertices at an interior point equals 360°. Since there are thirty such points, the sum of all the angles with vertices at these points equals $30 \cdot 360° = 10{,}800°$. As for the angles with vertices at the vertices of the 100-gon, their sum equals $(100 - 2) \cdot 180° = 17{,}640°$. Therefore the sum of the interior angles of all the triangles equals $10{,}800° + 17{,}640° = 28{,}440°$. Since the sum of the interior angles of a triangle is 180°, it follows that the total number of triangles equals

$$\frac{28{,}440}{180} = 158.$$

53. Construct the point B' symmetric to the point B with respect to the given line.

54. The frequency of occurrence of the letters in written English is important in a number of problems, e.g., in constructing optimum codes for telegraphic communication, in cryptanalysis, etc. Therefore experiments like the one suggested here, but involving much larger texts, are often carried out. It is generally accepted that the letters of the English language arranged in order of decreasing frequency are

$$e, t, r, i, n, o, a, s, d, l, c, h, f, u, p, m, y, g, w, v, b, x, k, q, j, z.$$

55. "I will not go so far as to say that to construct a history of thought without profound study of the mathematical ideas of successive epochs is like omitting Hamlet from the play which is named after him. That would be claiming too much. But it is certainly analogous to cutting out the part of Ophelia.

* Use the Euclidean algorithm (described in the answer to Prob. 122).

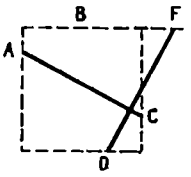

Fig. 23

This simile is singularly exact. For Ophelia is quite essential to the play, she is very charming and a little mad."

Whitehead

56. Seventy-five balls.

57. Let A, B, C and D be the given points (see Fig. 23). Draw the segment DF perpendicular to the segment AC and equal to it. Show that the point F lies on a side of the required square.

58. First transform the relation

$$m^2 + (m + 1)^2 = n^4 + (n + 1)^4$$

into

$$m(m + 1) = n(n + 1)[n(n + 1) + 2].$$

Then prove that the product of two consecutive positive integers cannot equal the product of two positive integers which differ by 2.

59. Since

$$m + (m + 1) + \cdots + (m + k) = 1000,$$

it follows from the formula for the sum of an arithmetic progression that

$$\frac{2m + k}{2}(k + 1) = 1000$$

or

$$(2m + k)(k + 1) = 2000.$$

Since

$$2m + k - (k + 1) = 2m - 1$$

is odd, one of the factors $2m + k$ and $k + 1$ is even and the other is odd. Moreover, obviously $2m + k > k + 1$. Therefore the problem has the following solutions:

$$2m + k = 2000,\ k + 1 = 1,\ m = 1000, k = 0;$$

$$2m + k = 400,\ k + 1 = 5,\ m = 198, k = 4;$$

$$2m + k = 125,\ k + 1 = 16, m = 55, k = 15;$$

$$2m + k = 80,\ k + 1 = 25, m = 28, k = 24.$$

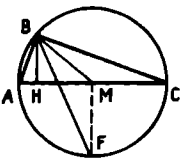

Fig. 24

60. Let ABC be the given triangle, BH the altitude, BM the median and BF the bisector extended until it intersects the circumscribed circle (see Fig. 24). Prove that BMF is an isosceles triangle. Deduce from this that B is a right angle.

61. Square the original inequality and use the fact that $2ab \leqslant a^2 + b^2$ for arbitrary a and b.

62. Note that every positive integer a can be written in the form $a = 2^k b$, where b is an odd number. Then note that among 101 odd numbers not exceeding 200, at least two must be equal.

63. Find numbers 10^{k_1} and 10^{k_2} which leave the same remainder after division by $9N$. The difference between 10^{k_1} and 10^{k_2} is a number made up entirely of nines and zeros. If this difference is divided by 9, we obtain a number made up entirely of ones and zeros which is divisible by N.

64. For example, one can use the theorem on the ratio of the segments into which the bisector divides the base.

65. Use a property of the median of a triangle.

66. Suppose N consists of n consecutive ones. Since N is divisible by

$$\underbrace{33\cdots33}_{\text{100 times}},$$

N is divisible by 3 and

$$\underbrace{11\cdots11}_{\text{100 times}}.$$

Therefore, besides being a multiple of 3, n must be a multiple of 100 (in this regard, see the answer to Prob. 51). The smallest number satisfying these requirements is $n = 300$. Therefore

$$N = \underbrace{11\cdots11}_{\text{300 times}}.$$

67. Suppose that after making some of the transformations described in the statement of the problem, the table T_1 goes into the table T_2 with the smallest number of minus signs, in the sense that further transformations of T_2 cannot decrease the number of minus signs. Such tables will be called *minimal.* As

we now show, the number of minuses in a minimal table cannot exceed four. In fact, there are no more than two minuses in each row and column of a minimal table (otherwise we could decrease the number of minuses by transforming the row or column). Suppose there exists a minimal table containing more than four minuses. Then some row A of the table contains two minuses. Let P and Q denote the columns containing these minuses. Two of the remaining minuses belong to the columns P and Q or to two other columns. By transforming the row A if necessary, we can arrange to have two minuses in each of two columns of the table. Now consider the fifth minus. It cannot belong to either of the columns just mentioned, since the table is minimal. Suppose this minus belongs to row B. Then by transforming one or two of the above-mentioned columns, we can arrange to have three minuses in a row. Contradiction!

The minimal tables are those with 0, 1, 2, 3 or 4 minuses on their diagonals (i.e., the characteristic can take the values 0, 1, 2, 3 and 4). To prove this, we note that the result of transforming the table T_1 into T_2 does not depend on the number of times a given row or column is changed, but rather on the parity of the number. For example, suppose the first row is changed a times, while the columns are changed b_1, b_2, b_3 and b_4 times, respectively. Then the sign in the upper left-hand corner is changed $a + b_1$ times, while the sign in the other three cells of the first row is changed $a + b_2$, $a + b_3$ and $a + b_4$ times, respectively. If a is even, it can be replaced by 0 without affecting the result, and similarly, if a is odd, it can be replaced by 1. Therefore we can confine ourselves to transformations in which each row and each column is changed no more than once. Using this fact, it is a simple matter to prove that the minimal tables are those with 0, 1, 2, 3 or 4 minuses on their diagonals.

68. Consider a convex n-gon no three diagonals of which intersect in a single point. The result of drawing all the diagonals is to divide the n-gon into a certain number of parts, a fact summarized by saying that the n-gon has been *partitioned.* Sup-

pose partitioning the n-gon gives P_3 triangles, P_4 quadrilaterals, ..., P_k k-gons. The problem is then to find the number $N = P_3 + P_4 + \cdots + P_k$.

First we calculate the total number of interior points of intersection of the diagonals of the n-gon. To this end, we note that any four vertices of the n-gon form a convex quadrilateral whose diagonals intersect inside the n-gon. Since four vertices can be selected from n vertices in

$$\frac{n(n-1)(n-2)(n-3)}{24}$$

distinct ways (verify this!), together the diagonals of the n-gon have

$$\frac{n(n-1)(n-2)(n-3)}{24}$$

interior points of intersection.

We now calculate the sum of all the angles of the subpolygons obtained by partitioning the n-gon. At every interior point of intersection of the diagonals there are four angles whose sum equals 360°, and moreover the angles whose vertices coincide with the vertices of the original n-gon have a sum equal to $(n-2)\cdot 180°$. It follows that

$$[P_3(3-2) + P_4(4-2) + \cdots + P_k(k-2)]\cdot 180°$$
$$= \frac{n(n-1)(n-2)(n-3)}{24}\cdot 360° + (n-2)\cdot 180°. \quad (1)$$

Next we find the total number of vertices of the subpolygons. Every interior point of intersection of the diagonals is a vertex of four subpolygons, while every vertex of the given n-gon is a vertex of $n-2$ subpolygons. Therefore

$$3P_3 + 4P_4 + \cdots + kP_k = \frac{n(n-1)(n-2)(n-3)}{24}\cdot 4 + n(n-2). \quad (2)$$

Formula (1) implies

$$3P_3 + 4P_4 + \cdots + kP_k - 2N = \frac{n(n-1)(n-2)(n-3)}{24} \cdot 2 + (n-2). \tag{3}$$

Comparing (3) and (2) and making some simple transformations, we finally obtain

$$N = \frac{(n-1)(n-2)(n^2-3n+12)}{24}.$$

In particular, if $n = 20$, then $N = 5016$.

69. Draw the diagonals of the square and use the Pythagorean theorem. The sum in question equals 1.

70.
$$\begin{aligned} x^8 + x^4 + 1 &= (x^8 + 2x^4 + 1) - x^4 \\ &= (x^4 + x^2 + 1)(x^4 - x^2 + 1) \\ &= (x^2 + x + 1)(x^2 - x + 1) \\ &\quad \times \left(x^2 + \sqrt{3}x + 1\right)\left(x^2 - \sqrt{3}x + 1\right). \end{aligned}$$

71. There are at least 97 students each of whom knows the other 99 students. To see this, we argue as follows: Suppose the 100 students do not all know each other. Then there are two students A and B who do not know each other. Consider any group of four students containing A and B, say A, B, X, Y. By hypothesis, one of the four knows the other three. Since A and B do not know each other, this student can only be X or Y. For example, suppose X knows A, B and Y. Applying the same argument to any group A, B, X, Z containing A, B and Z, we find that either X knows A, B and Z or else Z knows A, B and X so that X knows Z in any event. Therefore X knows all the members of the club. In other words, in any group of four students containing A and B, there is a student who knows all the members of the club. It follows that besides A and B, there is at most one more student who does not know all the members of the club.

72. If such a partition were possible, the total number of sides of black triangles would exceed the total number of sides of white triangles by 10. On the other hand, both of the numbers must be multiples of 3. But this is impossible.

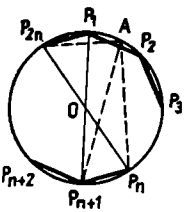

Fig. 25

73. We solve the problem for any regular polygon with an even number of sides. For each vertex of such a polygon, there is a diametrically opposite point (see Fig. 25). Therefore

$$AP_1^2 + AP_2^2 + \cdots + AP_{2n}^2$$
$$= (AP_1^2 + AP_{n+1}^2) + (AP_2^2 + AP_{n+2}^2) + \cdots + (AP_n^2 + AP_{2n}^2)$$
$$= (2R)^2 + (2R)^2 + \cdots + (2R)^2 = 4nR^2.$$

There is an analogous result for a regular n-gon with an odd number of sides, i.e., the sum of the squares of the chords joining a point of the circumscribed circle to the vertices of the n-gon equals $2nR^2$. However, this fact is harder to prove.

74. Since the product has eight digits, the number cannot terminate in a zero. But the product is divisible by 1000, so that one of the factors must be divisible by 125 while the other is divisible by 8. The numbers which are divisible by 125 and do not terminate in a zero must terminate in one of the following combinations of digits: 125, 375, 625, 875. Considering each case separately, we find that

$$abcd = 6125, 6375, 4625, 4875,$$
$$dcba = 5216, 5736, 5264, 5784.$$

Four more answers can be obtained by interchanging *abcd* and *dcba*.

75. In the triangle ABC extend the median AM past M by the segment $MD = AM$. Noting that $BD = AC$, construct the triangle ABM (see Fig. 26).

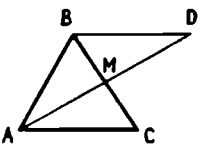

Fig. 26

76. Let A be the common point of the circles. Join A to the centers of the circles and consider the smallest of the angles so obtained. Show that the segment joining the corresponding centers O_1 and O_2 lies entirely in one disk.

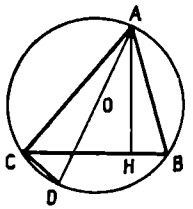

Fig. 27

77. Suppose the number m_1, made up of the given digits, is divisible by m_2 ($m_2 < m_1$). Then $m_1 - m_2$ is divisible by m_2. But $m_1 - m_2$ is divisible by 9 (see the hint to Prob. 39), while m_2 is not even divisible by 3. Therefore $m_1 - m_2$ is divisible by $9m_2$. But $9m_2$ is an eight-digit number. Contradiction!

78. Inscribe ABC in a circle and note that $\angle ADC = \angle ABC$ (see Fig. 27).

79. Consider the following three cases:

1) $x \leqslant 0$. Then each monomial appearing in the polynomial is nonnegative and hence

$$x^{12} - x^9 + x^4 - x + 1 \geqslant 1 > 0.$$

2) $0 < x < 1$. In this case

$$x^{12} - x^9 + x^4 - x + 1 = (1 - x) + x^4(1 - x^5) + x^{12} > 0.$$

3) $x \geqslant 1$. Then

$$x^{12} - x^9 + x^4 - x + 1 = x^9(x^3 - 1) + x(x^3 - 1) + 1$$
$$\geqslant 1 > 0.$$

80. Transforming the right-hand side of the given equation, we obtain

$$\begin{aligned}
&\tfrac{1}{101} + \cdots + \tfrac{1}{200} \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{199} + \tfrac{1}{200} - 1 - \tfrac{1}{2} - \cdots - \tfrac{1}{100} \\
&= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{199} + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \tfrac{1}{200} \\
&\quad - 2\left(\tfrac{1}{2} + \tfrac{1}{4} + \cdots + \tfrac{1}{200}\right) \\
&= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{199} - \tfrac{1}{2} - \tfrac{1}{4} - \cdots - \tfrac{1}{200}.
\end{aligned}$$

81. Ten ways.

82. Any longitudinal or transverse ruling divides the board into two regions, each containing an even number of squares. Of these, an even number (possibly zero) are covered by dominoes which do not intersect the boundary of the region. There-

fore the boundary can only be intersected by an even number of dominoes, say 0, 2, 4, or more. There are 102 such rulings, and hence at least 204 dominoes are needed to intersect them all. But there are only 200 dominoes, by hypothesis.

83. Rotate the triangle ACD about the point C by 60° so that the point D coincides with the point E.

84. Let A_1 denote an arbitrary student, and suppose he has solved problems a_1 and a_2. Then A_1 is asked to discuss problem a_2, say. Let A_2 be the unique student who in addition to A_1 has solved problem a_2. This student is then asked to discuss the other problem a_3 solved by him. Continue this process and show that sooner or later we call upon the student who in addition to A_1 has solved problem a_1. What should be done next if there are students who have not yet been called upon?

85. Let A_1 be a vertex of the decagon, and let M and N be the points of tangency of the sides A_1A_3 and A_1A_2 with the inscribed circle. Show that $\angle MON = 180° - \angle A_1$ (see Fig.28). Use this to construct a decagon similar to the given one.

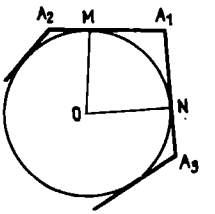

Fig.28

86. The number $\frac{1}{2}n(n+1)$ terminates in the same digit as $\frac{1}{2}(n+20)(n+21)$, since their difference is divisible by 10.

87. Represent each of the numbers written in the squares of the chessboard as a sum of two other numbers in the way shown in Figure 29. Since no two queens threaten each other, there is one queen in each horizontal and vertical line. Therefore the

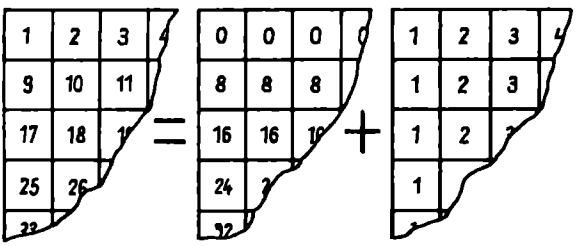

Fig.29

sum of the numbers in the squares occupied by the queens will always equal

$$(1 + 2 + 3 + \cdots + 8) + (0 + 8 + 16 + \cdots + 56) = 260.$$

88. Note that

$$(n + 1)(n + 2) \cdots (2n - 1)\, 2n$$

$$= \frac{1 \cdot 2 \cdots 2n}{1 \cdot 2 \cdots n} = 1 \cdot 3 \cdot 5 \cdots (2n - 1) \frac{2 \cdot 4 \cdots 2n}{1 \cdot 2 \cdots n}$$

$$= 1 \cdot 3 \cdot 5 \cdots (2n - 1)\, 2^n.$$

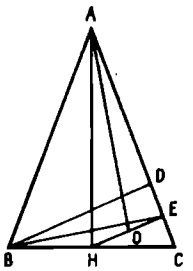

Fig. 30

89. Drop the perpendicular from B to AC (see Fig. 30). The triangles CBD, CEH and AHE are similar. The points E and O are midpoints of corresponding sides, and hence the triangles BEC and AHO are also similar. But AO is perpendicular to BE, since AH is perpendicular to BC.

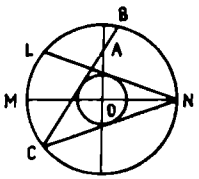

Fig. 31

90. Nothing changes if we assume that the given circle is of unit circumference. First we prove that a chord can always be drawn through the point A subtending an arc of rational length p/q. To see this, draw the diameter MN perpendicular to the segment OA (see Fig. 31), and then from the point N mark off an arc NL of rational length p/q so close to $\frac{1}{2}$ that the chord LN passes between the points A and O. Next draw the circle with center O tangent to LN, and construct the chord CB passing through A and tangent to this circle. The arc CB has the same rational length p/q as the arc NL.

If we now direct a ray of light from the point A along the chord BC, then after each reflection the ray advances by an arc equal to p/q. Therefore after q reflections (or possibly fewer) the ray returns to the original point (verify this and draw an appropriate figure). Thus the problem is always solvable and in fact has infinitely many solutions.

91. First note that each segment of the polygonal curve can intersect no more than 200 segments (the segment clearly cannot

intersect itself or the adjacent two segments). Therefore the total number of points of self-intersection cannot exceed

$$\frac{203 \cdot 200}{2} = 20{,}300.$$

For example, a regular star-shaped 203-gon has precisely 20,300 points of self-intersection. How it should be drawn is apparent from Figure 32, which shows a regular star-shaped heptagon. Convince yourself that no three segments of a regular star-shaped 203-gon intersect in a single point.

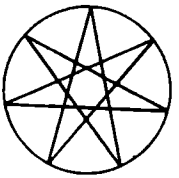

Fig. 32

92. Denote the circles by C_1, C_2, C_3, C_4 and C_5. Let A be the common point of the circles C_1, C_2, C_3, and C_4, B the common point of C_1, C_2, C_3 and C_5, and C the common point of C_1, C_2, C_4 and C_5. If A, B and C are distinct, the circles C_1 and C_2 have three points in common and hence coincide. But then the common point of the circles C_1, C_3, C_4 and C_5 also belongs to C_2. If, on the other hand, the points A and C (say) coincide, then A is a common point of all the circles.

93. The pedestrians are closest together when they are at the points C and D equidistant from the point O (see Fig. 33). To see this, suppose they are at two other points E and F. Then constructing the segment DE' symmetric to DE with respect to CD, we find that $EF > FE' = CD$ (since EF is the hypotenuse of the right triangle $FE'E$). If $\angle AOB = 90°$, $AO = a$, $BO = b$ and $a > b$, then the minimum distance between the pedestrians equals $\dfrac{a-b}{\sqrt{2}}$ and is reached at the time $\dfrac{a+b}{2v}$ after the motion begins.

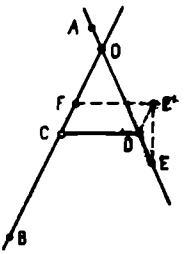

Fig. 33

94. Since the triangles ABC and ACD have the same area (see Fig. 34), they have equal altitudes BE and DF. Therefore the quadrilateral $BEDF$ is a parallelogram and its diagonals are divided in half by the point of intersection O, i.e., $BO = OD$. The fact that $AO = OC$ is proved in just the same way. Therefore the diagonals of the quadrilateral $ABCD$ are divided in half by their point of intersection, i.e., $ABCD$ is a parallelogram.

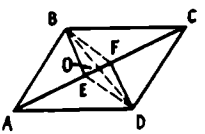

Fig. 34

95. It is easy to see that any two points in Figure 7 can be joined by a polygonal curve containing no more than 5 segments. Consider such a curve joining the points labelled 1 and 18, and show that the numbers at the end points of one of its segments differ by more than 3.

96. If $m > n$, factor $2^{2^m} - 1$ and prove that it is divisible by $2^{2^n} + 1$.

97. One solution can be obtained by drawing a line parallel to the base such that the altitudes of the two resulting trapezoids are proportional to their bases.

98. If the given inequalities hold, then $a \leqslant 1$, $b \leqslant 1$, $c \leqslant 1$. Prove that $x(1 - x) \leqslant \frac{1}{4}$ for arbitrary x, which implies

$$a(a - b)\, b(1 - c)\, c(1 - a) \leqslant \tfrac{1}{64}.$$

But multiplying the given inequalities together (why is this possible?), we obtain

$$a(1 - b)\, b(1 - c)\, c(1 - a) > \tfrac{1}{64}.$$

Contradiction!

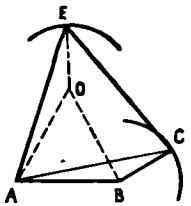

Fig. 35

99. Expand the left-hand side and use the inequality $a^2 + b^2 \geqslant 2ab$ several times.

100. As the rod BC is rotated about B, the point C describes a circle. Using the fact that the point E can be obtained by rotating AC about A through 60°, prove that the point E also describes a circle. Then on this circle find the point furthest from B (see Fig. 35). It turns out that the maximum value of BE equals $a + b$. In this case, $\angle ABC = 120°$.

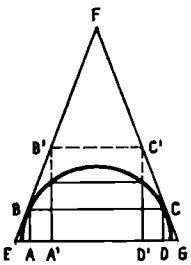

Fig. 36

101. First prove that if a rectangle is inscribed in a semicircle, then there is a rectangle with the same or even a larger perimeter whose lower base lies on the diameter. Now suppose the side AD of the rectangle $ABCD$ is four times larger than AB. Through B and C draw the tangents to the semicircle and prove that all the rectangles inscribed in the triangle EFG have the same perimeter (see Fig. 36).

102. Suppose $AB > AC$. Apply a triangle $AB'C'$ similar to

ABC and such that $AC < AB' < AB$ to the side AC of the triangle ABC in the way shown in Figure 37. Then draw a line DE parallel to AC through the point of intersection E of the lines BC and $B'C'$. The triangles $AB'C$ and DBE satisfy the conditions of the problem.

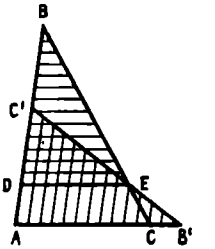

Fig. 37

103. Through the point P draw the segment $A'B'$ equal and parallel to AB (see Fig. 38). Then apply the parallelogram $AA'B'B$ to the side CD in such a way that the point A coincides with C and the point B coincides with D.

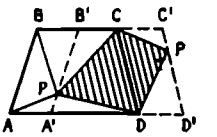

Fig. 38

104. Suppose that in every position fewer than 100 sectors of the inner disk lie on top of identically colored sectors of the outer disk. Then the total number of cases in which the colors of the sectors coincide is less than $100 \cdot 200$. On the other hand, if the outer disk is given a complete rotation through 360°, every sector of the inner disk will be covered 100 times by a sector of the same color and 100 times by sectors of the opposite color. Therefore the total number of cases where superimposed sectors of the inner and outer disks have the same color is precisely $100 \cdot 200$. Contradiction!

105. Transform the system into

$$x + y = a - z,$$

$$\frac{a - z}{xy} = -\frac{a - z}{az}.$$

Suppose $a \neq z$. Then, dividing by $a - z$, we obtain

$$x + y = a + (-z),$$

$$xy = a(-z).$$

It follows that the numbers a and $-z$ are roots of the same quadratic equation as the numbers x and y (why?). Therefore one of the numbers x and y coincides with a.

106. There are 9 one-digit numbers, $99 - 9 = 90$ two-digit numbers, $999 - 99 = 900$ three-digit numbers, and in general $9 \cdot 10^{n-1}$ n-digit numbers. The nine-digit numbers occupy

9 places in the given sequence, the two-digit numbers occupy 180 places, the three-digit numbers $900 \cdot 3 = 2700$ places, the four-digit numbers $9000 \cdot 4 = 36000$ places and the five-digit numbers $90000 \cdot 5 = 450000$ places. It follows that the required digit belongs to a five-digit number.

The digits belonging to numbers with no more than four digits belong to the places from 1 to $9 + 180 + 2700 + 36000 = 38889$. To find out how many five-digit numbers lie between the 38889th place and the 206788th place, we divide the difference $206788 - 38889 = 167899$ by 5, obtaining

$$206788 - 38889 = 5 \cdot 33579 + 4.$$

Therefore the required digit belongs to the 33580th five-digit number, i.e., to the number 43579 (since the first five-digit number is 10000). Moreover, the digit occupies the fourth place in this number and hence equals 7.

107. Each of the 25 numbers appears 25 times in the table and an even number of times off the main diagonal. Hence each of the numbers appears an odd number of times on the main diagonal. This is possible only if each number appears just once on the main diagonal.

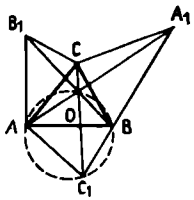

Fig. 39

108. On the sides of the triangle ABC construct the similar triangles ABC_1, BCA_1, CAB_1 with angles $180° - \angle H$, $180° - \angle K$, $180° - \angle M$ (see Fig. 39). Show that the point O lies at the intersection of the lines CC_1 and BB_1. To see this, verify that the point O lies on the circles circumscribed about the triangles ABC_1 and AB_1C.

109. Rotate the triangle ABN about the point B through 90° so that BN coincides with BK, and prove that $CBA'K$ is a triangle with BE' joining the midpoints of two of its sides. Deduce from this that $\angle FBE = 180°$ (see Fig. 40).

110. Note that there 25 tiles in all, each of which covers either three white squares or three black squares.

111. Yes. Find the two points A and B which are closest together, and consider the smallest of the circles passing through

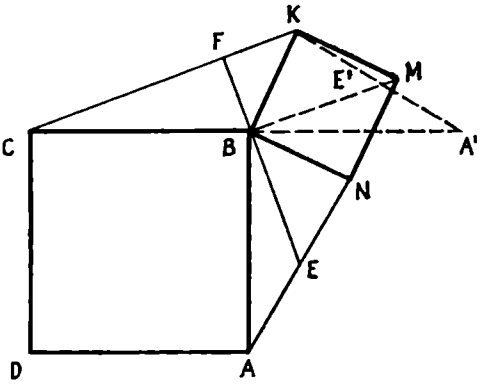

Fig. 40

A and B and each of the remaining $n - 2$ points. Then none of the n points lies inside this circle (why?).

112. Let the squares in the upper and lower rows be called *extreme* squares, and let the others be called *intermediate* squares. The knight can arrive at an extreme square only by leaving an intermediate square, and hence to traverse all the squares of the board in the way required, the knight must make $2n$ moves from intermediate squares to extreme squares. Clearly the remaining moves must be made from extreme squares to intermediate squares. But the knight always moves from a square of a given color to a square of the opposite color. It follows that the extreme squares are all of one color, while the intermediate squares are all of the opposite color. Contradiction!

113. The proof is by induction. For $n = 4$ the assertion is obviously true. Suppose it is true for $n = k$, and choose any k points from a set of $k + 1$ points satisfying the condition of the problem. By the induction hypothesis, these points lie at the vertices of a convex k-gon. The $(k + 1)$st point cannot lie inside this k-gon, as can be seen at once by drawing the diagonals emanating from one vertex. Suppose we extend the sides of the k-gon as shown in Figure 41, corresponding to the case $k = 5$

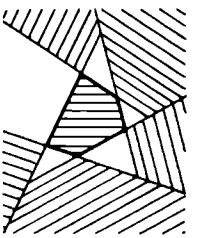

Fig. 41

(for simplicity). Then the $(k + 1)$st point cannot lie in any of the shaded regions (why not?). However, if it lies in one of the unshaded regions, then all $k + 1$ points lie at the vertices of a convex $(k + 1)$-gon. The induction is now complete.

114. Prove that the cube of a positive integer can be written in the form $9k$, $9k + 1$ or $9k - 1$, where k is itself a positive integer. Deduce from this that numbers of the form $9k + 4$ and $9k + 5$ cannot be written as sums of cubes of three positive integers.

115. Note that

$$10^n - 1 = \underbrace{99 \cdots 99}_{n \text{ times}} = 9 \times \underbrace{11 \cdots 11}_{n \text{ times}},$$

and hence

$$10^n + 18^n - 1 = 9(\underbrace{11 \cdots 11}_{n \text{ times}} + 2n).$$

But the number in parentheses is divisible by 3. To see this, we use the fact that the sum of the digits of a number leaves the same remainder upon division by 3 as the number itself (why?). Suppose dividing n by 3 gives the remainder α, so that $n = 3k + \alpha$. Then, since the sum of the digits of the number $\underbrace{11 \cdots 11}_{n \text{ times}}$ equals n, we have

$$\underbrace{11 \cdots 11}_{n \text{ times}} = 3l + \alpha$$

and hence

$$\underbrace{11 \cdots 11}_{n \text{ times}} + 2n = 3(l + 2k + \alpha).$$

116. Complete the triangle ABC to a parallelogram and prove that $2m_a < b + c$, where m_a is the median drawn to the side a, and b and c are the other sides of the triangle.

117. The problem has the following solutions:

3 080 103, 3 020 303, 3 060 603, 3 000 803, 3 050 203, 3 090 503, 3 030 703.

118. Note that

$$a + a^2 + \cdots + a^{100} = a^{101}(a^{-1} + a^{-2} + \cdots + a^{-100}).$$

119. Let $10a + b$ be the required number, where a and b are positive integers not exceeding 9. Then we must solve the equation

$$(10a + b)^2 = (a + b)^3. \tag{1}$$

Both sides of (1) determine a certain number N which is both a perfect square and a perfect cube. Suppose the prime number p appears to the kth power in the expansion of N into prime factors. Then k must be divisible by both 2 and 3, and hence by 6 as well. Therefore $N = n^6$, where n is a positive integer. It follows that $a + b$ must be a perfect square. But $a + b \leqslant 18$ and hence $a + b$ can only take the values 1, 4, 9, 16. Similarly, we find that $10a + b$ must be a perfect cube and hence can only take the values 1, 8, 27, 64. Then, by direct substitution, we find that the unique solution of the problem is $10a + b = 27$.

An alternative approach to the problem is the following: Writing the required number as $10a + b = 9a + (a + b)$, transform equation (1) into

$$81a^2 = (a + b)[(a + b)(a + b - 1) - 18a].$$

Deduce that $a + b$ is divisible by 9. Then consider the two cases $a + b = 9$ and $a + b = 18$, and verify that 27 is the unique solution of the problem.

120. Multiply the first equation by y and subtract it from the second equation.

121. Let a be the quotient and b the remainder obtained when the prime number p is divided by 30, so that $p = 30a + b$. Suppose b is a composite number. Since any composite number less than 30 shares a common factor $q > 1$ with 30 (verify this!), it follows that $p = 30a + b$ is divisible by q contrary to the assumption that p is prime.

122. Let a_n denote the nth term of the Fibonacci sequence.

Then, by definition,

$$a_1 = 1, a_2 = 1, a_3 = 2, \ldots, a_{n+2} = a_n + a_{n+1}, \ldots$$

Setting $a_0 = 0$, we also have $a_2 = a_0 + a_1$. The terms of the Fibonacci sequence are called the *Fibonacci numbers* and have the following interesting properties:

1) *For arbitrary k and l,*

$$a_{k+l} = a_k a_{l-1} + a_{k+1} a_l. \tag{1}$$

To prove (1), we use induction in k. The identity is trivial for $k = 0$, while for $k = 1$ it reduces to the basic relation $a_{l+1} = a_{l-1} + a_l$. Suppose (1) holds for some $k - 1$ and k. Then

$$\begin{aligned} a_{k+1+l} &= a_{k+l} + a_{k+l-1} \\ &= a_k a_{l-1} + a_{k+1} a_l + a_{k-1} a_{l-1} + a_k a_l \\ &= a_{l-1}(a_{k-1} + a_k) + a_l(a_k + a_{k+1}) \\ &= a_{l-1} a_{k+1} + a_l a_{k+2}, \end{aligned}$$

i.e., (1) holds for $k + 1$ also and the induction is complete.

2) *For arbitrary k and n, a_{kn} is divisible by a_k.* This property is easily proved by induction in n with the help of formula (1).

3) *Two consecutive Fibonacci numbers are always relatively prime.* In fact, suppose a_k and a_{k+1} have a common divisor $l \neq 1$. Then it follows from $a_{k-1} = a_{k+1} - a_k$ that a_{k-1} is also divisible by l. But since a_k and a_{k-1} are divisible by l, so is a_{k-2}. Repeating this argument, we eventually find that the numbers a_1 and a_2 are both divisible by l, which is impossible.

We are now in a position to find the greatest common divisor of the Fibonacci numbers a_{770} and a_{1000}. By property 1,

$$a_{1000} = a_{770} a_{229} + a_{771} a_{230}. \tag{2}$$

If a_{770} and a_{1000} have a common divisor r, then, by property 3, a_{230} is also divisible by r. Moreover, by property 1 again,

$$a_{770} = a_{690}a_{79} + a_{691}a_{80}. \tag{3}$$

By property 2, a_{690} is divisible by a_{230} and hence also by r. Therefore, as before, we find that a_{80} is divisible by r. In just the same way, it follows from

$$a_{230} = a_{160}a_{69} + a_{161}a_{70} \tag{4}$$

that a_{70} is divisible by r and from the relation

$$a_{80} = a_{70}a_9 + a_{71}a_{10} \tag{5}$$

that a_{10} is divisible by r. By property 2, a_{10} is a common divisor of a_{770} and a_{1000}. But, as just shown, a_{10} is divisible by any common divisor r of the numbers a_{770} and a_{1000}. It follows that a_{10} is the greatest common divisor of a_{770} and a_{1000}.

Similarly, it can be shown that the greatest common divisor of the Fibonacci numbers a_m and a_n equals a_s, where s is the greatest common divisor of m and n. To prove this, we start from the chain of equations used to find the greatest common divisor of the numbers m and n, which in the case of 770 and 1000 take the form*

$$\begin{aligned} 1000 &= 770 \cdot 1 + 230, \\ 770 &= 230 \cdot 3 + 80, \\ 230 &= 80 \cdot 2 + 70, \\ 80 &= 70 \cdot 1 + 10. \end{aligned} \tag{6}$$

Note the connection between these equations and formulas (2) through (5).

123. Let a and b be the first and last digits of the number in question. The number itself can then be written in the form

* Prove that the last nonzero remainder in a chain of equations like (6) is the greatest common divisor of the original numbers, here 770 and 1000 (this is the so-called *Euclidean algorithm.*).

$a \cdots b$, and the reverse of the number takes the form $b \cdots a$. By hypothesis,

$$\frac{a \cdots b}{b \cdots a} = k,$$

where k is a positive integer distinct from 1. The sense of the problem requires that both $a \neq 0$ and $b \neq 0$ (if the hypothesis $b \neq 0$ is dropped, the problem has very many solutions which it would be hard to describe). Moreover, it is clear that $k < 10$. We now explore the connection between a and b.

First of all, it is clear from the relation

$$b \cdots a = \frac{a \cdots b}{k}$$

that $b = \left[\frac{a}{k}\right]$.* Secondly, the last digit of the product ak equals b. Since $b \neq 0$, it follows that $a \geqslant k$ $\left(\text{otherwise } \left[\frac{a}{k}\right] = 0\right)$. This prompts us to construct Table 1, each cell of which contains first $\left[\frac{a}{k}\right]$ and then the last digit of the product of a and k. Here we are interested only in values of a and k such that the two digits appearing in the corresponding cell are equal. As shown by Table 1, the problem has solutions only if $a = 8$, $k = 4$, $b = \left[\frac{a}{k}\right] = 2$ or if $a = 9, k = 9, b = \left[\frac{9}{9}\right] = 1$. We now consider each of these cases in turn.

Suppose $a = 8$, $b = 2$, $k = 4$ and let c and d denote the second and the next-to-the-last digits of the required number, so that

$$\frac{8c \cdots d2}{2d \cdots c8} = 4.$$

Then it is not hard to see that $d = \left[\frac{c}{4}\right]$ and that the last digit

* Here and henceforth, $[x]$ means the *integral part* of $[x]$, i.e., the largest positive integer not exceeding x.

a	k							
	2	3	4	5	6	7	8	9
2	1 4							
3	1 6	1 9						
4	2 8	1 2	1 6					
5	2 0	1 5	1 0	1 5				
6	3 2	2 8	1 4	1 0	1 6			
7	3 4	2 1	1 8	1 5	1 2	1 9		
8	4 6	2 4	2 2	1 0	1 8	1 6	1 4	
9	4 8	3 7	2 6	1 5	1 4	1 3	1 2	1 1

Table 1

of the number $4c + 3$ also equals d. Giving c various values, we construct Table 2, which shows that $c = 7$, $d = 1$.

Next, denoting the third digit by e and the third digit from the end by f, we find that

$$\frac{87e \cdots f12}{21f \cdots e78} = 4.$$

It follows that $f = \left[\frac{30 + e}{4}\right]$ and that f is the last digit of $4e + 3$. This in turn leads to Table 3.

c	0	1	2	3	4	5	6	7	8	9
$\left[\frac{c}{4}\right]$	0	0	0	0	1	1	1	1	2	2
Last digit of $4c + 3$	3	7	1	5	9	3	7	1	5	9

Table 2

e	0	1	2	3	4	5	6	7	8	9
$\left[\frac{30 + e}{4}\right]$	7	7	8	8	8	8	9	9	9	9
Last digit of $4e + f$	3	7	1	5	9	3	7	1	5	9

Table 3

Two combinations of digits satisfy the conditions of the problem, namely $e = 1$, $f = 7$ and $e = 9$, $f = 9$. In the first case, an entirely analogous argument shows that the reverse of the required number is of the form $2178 \cdots 2178$. Moreover, since $2178 \cdot 4 = 8712$, the number $2178 \cdots 2178$ satisfies the conditions of the problem if and only if the same is true of the number corresponding to the missing digits (here, of course, runs of zeros are allowed). In the second case ($e = 9$, $f = 9$), we have

$$\frac{879g \cdots h912}{219h \cdots g978} = 4.$$

It is easy to see that $h = \left[\frac{30 + g}{4}\right]$ and that h equals the last digit of $4g + 3$. Therefore the possibilities for g and h are the same as for e and f, i.e., $g = 1$, $h = 7$ or $g = 9$, $h = 9$. These

considerations show that the required number can take any of the following forms:

$$8712,\ 87912,\ 879912,\ \ldots,\ \underbrace{8799\cdots9912}_{s\text{ times}},\ \ldots$$

If a number begins with one of these combinations, it must end in the same combination, and the number corresponding to the intermediate digits must also be a solution of the problem. For example, the number

$$8\,791\,200\,871\,287\,120\,087\,912$$

equals four times the same number written in reverse order.

This completes the analysis of the case $a = 8$, $b = 2$, $k = 4$. The analysis of the case $a = 9$, $b = 1$, $k = 9$ is completely analogous and leads to the following simple combinations:

$$9801,\ 98901,\ 989901,\ \ldots,\ \underbrace{9899\cdots9901}_{s\text{ times}},\ \ldots$$

For example, the number

$$9\,899\,901\,000\,980\,100\,009\,899\,901$$

equals nine times the same number written in reverse order.

124. $(a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + \cdots$

$$+ (k-1)(a_{k-1} - a_k) + ka_k$$

$$= a_1 + a_2 + \cdots + a_k.$$

125. $N = 2^{15}3^{10}5^6$.

126. First note that $2? = 2$. If $n > 2$, then $n? > n$. In fact, consider the number

$$n? - 1 = 2 \cdot 3 \cdot 5 \cdots p - 1,$$

where p is the largest prime number not exceeding n. This number is not divisible by any prime number less than or equal to n, i.e., it is either prime itself or else it is divisible by a

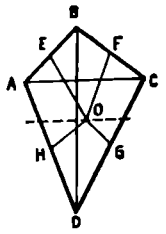

Fig. 42

prime number greater than p and hence greater than n. Therefore $n? - 1 > n$ and $n? > n$.

127. Let O be any point of the line passing through the midpoint of the diagonal BD parallel to AC (see Fig. 42). Prove that the area of each quadrilateral $HOGD$ and $EOFB$ equals one fourth the area of $ABCD$.

128. Note that the point M can be obtained by rotating the segment PK about P through 60°. The figure in question is the square obtained by rotating Q about P through 60°.

129. First we prove the following lemma:

Lemma. The locus of the points whose distances from two given points A and B have a constant ratio k is a circle (called the circle of Apollonius).

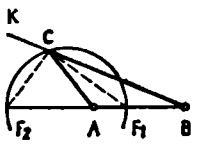

Fig. 43

Proof. Let C be a point of the locus in question, and draw the bisector CF_1 of the angle ACB (see Fig. 43). By a familiar theorem,

$$\frac{F_1A}{F_2B} = \frac{CA}{CB} = k.$$

Therefore the bisector of the angle ACB always goes through the fixed point F_1. Similarly, the bisector of the exterior angle KCA always goes through the point F_2 such that

$$\frac{F_2A}{F_2B} = \frac{CA}{CB} = k$$

(prove this!). But F_2CF_1 is a right angle. Hence the required locus is the circle with diameter F_1F_2.

Using the lemma, we can now easily solve the problem. In fact, if O is the required point (see Fig. 44), then

$$\frac{CO}{OB} = \frac{DO}{OA} = \frac{CD}{AB} = k,$$

and hence O lies at the intersection of the circles of Apollonius constructed for the pairs of points D, A and C, B with the given ratio $k = CD/AB$.

The problem has four solutions.

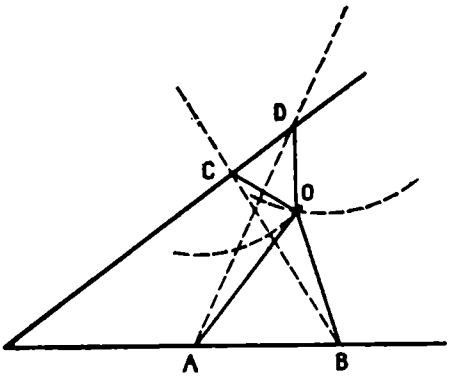

Fig. 44

130. Take the row for which the sum of the numbers is the smallest, and consider the sum of the numbers in all the "crosses" corresponding to the cells in this row. The sum of all these sums is no less than na. On the other hand, denoting the sum of all the numbers in the table by N and the sum of the numbers in this row by m, we find that

$$N + (n - 1)\, m \geqslant na.$$

But $m \leqslant N/n$ by hypothesis, and hence

$$N\left(1 + \frac{n-1}{n}\right) \geqslant an$$

or

$$N \geqslant \frac{an^2}{2n-1}.$$

Therefore the sum of the numbers in the table is no less than $an^2/(2n - 1)$. It only remains to show that N is precisely $an^2/(2n - 1)$ for some choice of numbers in the table (do so!).

131. Suppose the other lines do not all go through the point O in which two given lines intersect. For example, suppose the line l does not go through O. Among the points of intersection not on l, choose the point A closest to l (see Fig. 45). By

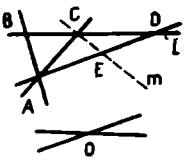

Fig. 45

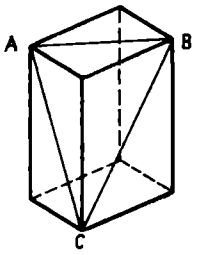

Fig. 46

hypothesis, at least three lines go through A. Suppose these lines intersect l in points B, C and D, where C lies between B and D. Besides l and AC, there must be another line m passing through C. This line intersects one side of the triangle ABD in a point E closer to l than A, contrary to the assumption that A is the point of intersection closest to l.

132. Prove that the area of the whole shadow equals twice the area of the shadow of the triangle ABC (see Fig. 46), so that to obtain the shadow of largest area, the match box should be held with the plane ABC horizontal. In the case of the triangular pyramid, the plane going through the midpoints of four edges intersecting two at a time must be horizontal (see Fig. 47). This can be accomplished by holding horizontal both a given edge and the edge opposite it. It should be noted that by casting shadows of three-dimensional objects on a plane surface, we are in effect finding their projections.

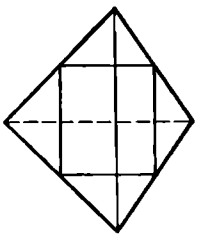
Fig. 47

133. Suppose the numbers a_1, a_2, a_3, a_4, a_5 and a_6 are written on the faces of the cube (see Fig. 48). Carrying out the operation described in the statement of the problem, we write the resulting numbers on the face of a second cube. Then we do the same thing with a third cube, and so on. As a result, we obtain 26 cubes such that the numbers written on the tenth cube, say, are obtained from those written on the ninth cube by carrying out the operation described in the statement of the problem. Let M_i $(i = 1, 2, \ldots, 26)$ denote the largest of the numbers written on the faces of the ith cube, and show that

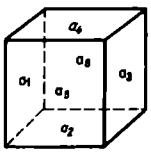
Fig. 48

$$M_1 \geqslant M_2 \geqslant M_3 \geqslant \cdots \geqslant M_{26}.$$

But by hypothesis $M_{26} = M_1$, and hence

$$M_1 = M_2 = M_3 = \cdots = M_{26}.$$

Now consider the third cube. One of its faces must be labelled with the number M_1. Suppose this is the upper face. By the argument given in the answer to Problem 48, it is easy to see that M_1 must appear on four adjacent faces of the second cube.

In just the same way, this implies that M_1 is written on all the faces of the first cube. But, by hypothesis, the numbers written on the faces of the first cube are not all equal. This contradiction shows that there must be a mistake in the calculations.

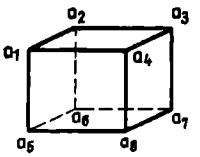

Fig. 49

134. Suppose the vertices of the cube are labelled with the numbers a_1 through a_8 (see Fig. 49). Carrying out the operation described in the statement of the problem, we write the resulting numbers at the vertices of a second cube. Then we do the same thing with a third cube, and so on. Let m_i and M_i $(i = 1, 2, \ldots, 11)$ denote the smallest and largest of the numbers labelling the vertices of the ith cube. By the same argument as in the preceding problem, prove that

$$m_1 = m_2 = \cdots = m_{11}, M_1 = M_2 = \cdots = M_{11}.$$

Finally, by the argument given in the answer to Problem 48, prove that $m_1 = 0$, $M_1 = 1$ and that the numbers originally at the vertices are those shown in Figure 50.

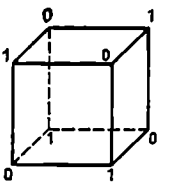

Fig. 50

135. The sum of the distances to the three sides of a triangle is smallest at one vertex of the triangle, namely the one belonging to the shortest altitude. The same sum is largest for the vertex belonging to the longest altitude. The problem does not always have a unique solution (consider isosceles and equilateral triangles).

136. The problem can be solved by simply enumerating all the possibilities. However, the amount of work is greatly reduced by making two observations:

1) The largest of the fractions $1/k$, $1/l$, $1/m$ can only equal $\frac{1}{3}$, $\frac{1}{4}$ or $\frac{1}{5}$.

2) The fraction $1/r$ is the best underestimate of the fraction $1/s$ for $r = s + 1$.

The three-term sum underestimating $\frac{1}{2}$ but as close to $\frac{1}{2}$ as possible is

$$\tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{43}.$$

Similarly, the four-term sum underestimating 1 but as close to 1 as possible is

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{43}.$$

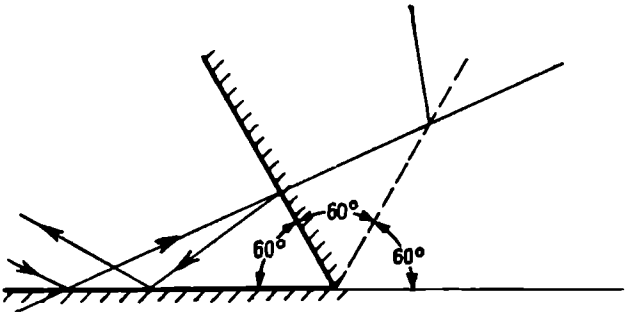

Fig. 51

137. Use the fact that after reflection from a line, a ray moves in such a way that its mirror image lies on the continuation of the path of the ray before reflection. The solution for the case of an angle of 60° is shown in Figure 51. A ray can be reflected 180 times from the sides of an angle of 1°.

138. Use the answer to the preceding problem. Prove that if a, b and c are the numbers of reflections of the ray from the three sides of the triangle, then

$$c - 1 \leqslant a + b \leqslant 3c + 3.$$

The numbers x and y can take any values such that the triple x, y, 4 satisfies this inequality (supply the details!).

139. First show that to return to its original position, the lion must make the same number of moves of all three kinds. But there are 100 squares in the board, and hence the lion cannot traverse the whole board and return to its original position.

140. Suppose White cannot guarantee a win or a draw. This means that regardless of how White plays, Black always wins (if he plays properly). But White can always exchange roles with Black (in effect), by moving out a knight on his first move and then returning the knight to its original position on his second move.

141. In the coordinate system shown in Figure 52a, the center of every tree has integral coordinates x and y satisfying the

inequality $x^2 + y^2 \leqslant s^2$. A point with integral coordinates will be called *primitive* if it is visible from the origin. It is not hard to see that a point (p, q) is primitive if and only if p and q are relatively prime. We now prove two preliminary results:

Lemma 1. If the numbers p and q are relatively prime, then there are integers u_0 and v_0 such that $pv_0 - qu_0 = 1$. Moreover, the general solution of the equation $pv - qu = 1$ is of the form $u = u_0 + kp$, $v = v_0 + kq$, where k is an arbitrary integer.

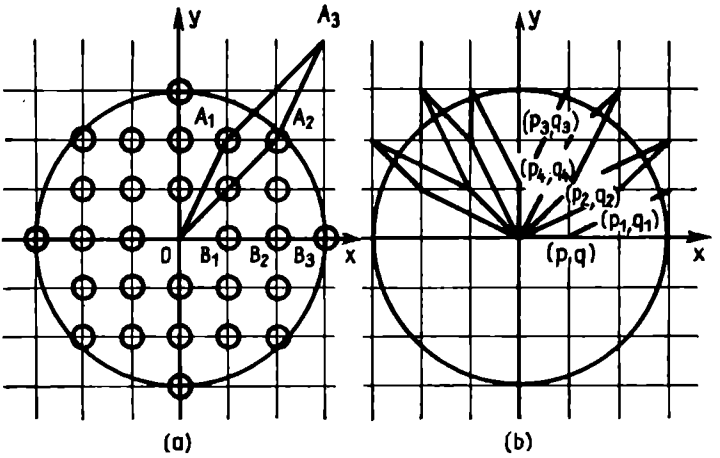

Fig. 52

Hint. Use the Euclidean algorithm (see the answers to Probs. 51 and 122).

Lemma 2. The area of the parallelogram spanned by the points (p, q) and (u, v) equals $pv - qu$.

Hint. Note that

$$\text{Area } (OA_1A_3A_2) = \text{Area } (OA_1B_1) + \text{Area } (B_1A_1A_3B_3) - \text{Area } (OB_2A_2) - \text{Area } (B_2A_2A_3B_3).$$

If $pv - qu = 1$, then u and v are also relatively prime. The point (u, v) will be called a "slant neighbor" of the point (p, q) and vice versa. Every primitive point (p, q) has infinitely many

slant neighbors all lying along one line at equal distances from one another.

If the points (p, q) and (u, v) are slant neighbors, then, by Lemma 2, the area of the parallelogram spanned by them equals 1. Suppose the diagonal of this parallelogram is of length d. Then (p, q) and (u, v) are both at distance $1/d$ from the diagonal. (From now on, a "diagonal" will always be taken to mean the diagonal drawn from the origin.)

Now let ϱ denote the smallest value of the radius of the trees such that the stand of trees lets no light through. We must prove that

$$\frac{1}{\sqrt{s^2 + 1}} \leqslant \varrho \leqslant \frac{1}{s}.$$

This will be done in two steps:

1) The points $(1, 0)$ and $(s - 1, 1)$ are slant neighbors, spanning a parallelogram whose diagonal is of length $\sqrt{s^2 + 1}$. This diagonal, when extended, can be blocked only by trees of radius ϱ with centers at the points $(1, 0)$ or $(s - 1, 1)$, and hence

$$\varrho \geqslant \frac{1}{\sqrt{s^2 + 1}}.$$

2) Given a primitive point (p, q) lying in the disk $x^2 + y^2 \leqslant s^2$, let (p_1, q_1) be its "extreme slant neighbor" in the sense that the point $(p_1 + p, q_1 + q)$ no longer lies in the disk $x^2 + y^2 \leqslant s^2$. In the same way, let (p_2, q_2) be the extreme slant neighbor of the point (p_1, q_1), (p_3, q_3) the extreme slant neighbor of (p_2, q_2), and so on. After a finite number of steps, we arrive at a point (p_n, q_n) such that the parallelograms spanned by (p, q) and (p_1, q_1), by (p_2, q_2) and (p_1, q_1), etc., and finally by (p_{n-1}, q_{n-1}) and (p_n, q_n) completely cover the disk $x^2 + y^2 \leqslant 1$ of unit radius (see Fig. 52b where $s = 3$ and $(p, q) = (1, 0)$). The diagonal of each of these parallelograms exceeds s, while the distance between the diagonal and the vertices not on the diagonal

is less than $1/s$. Therefore trees of radius $1/s$ with centers at the points (p, q), (p_1, q_1), ..., (p_n, q_n) block any ray of light leaving the origin, and hence

$$\varrho \leqslant \frac{1}{s}.$$

142. There is no loss of generality in making three simplifying assumptions:

1) *No more than three countries meet at any given vertex.* To see this, consider a vertex at which more than three countries meet, like the one shown in Figure 53a. Suppose we introduce a small new country near this vertex, in the way shown in Figure 53b. Then the new map contains an extra country,

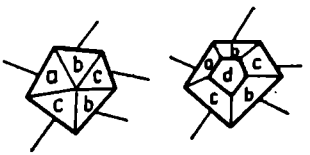

Fig. 53

but we have eliminated a vertex at which more than three countries meet. Doing the same thing for all such vertices, we finally obtain a map containing no vertices at which more than three countries meet. But if 12 colors suffice to color the new map, they are obviously enough to color the old map. In fact, we need only give each country in Figure 53a the same color as it has in Figure 53b.

2) *There are no countries with ring-shaped parts.* If country *a* has a ring-shaped part as shown in Figure 54, and if country *b* is adjacent to the inner boundary of the ring, then together with *b* consider some part *c* of country *a* and some country *d* adjacent to the outer boundary of the ring. If 12 colors suffice to color the new map, they are certainly enough to color the old map.

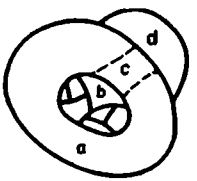

Fig. 54

3) *Each country has precisely two parts.* In fact, if country e has only one part, we merely give e a small extra piece of territory near a vertex where three other countries, say f, g and h, meet (see Fig. 55). If the new map can be colored with 12 colors, then obviously so can the old map.

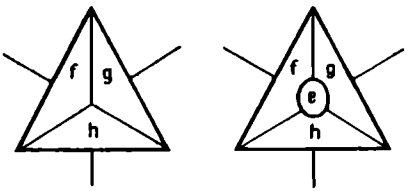

Fig. 55

Now suppose the map has c_4 countries with four vertices, c_5 countries with five vertices, and so on (we have in mind the number of vertices of both parts of the country). The total number of countries is then

$$c = c_4 + c_5 + c_6 + \cdots.$$

Each country with four vertices has four boundaries, each country with five vertices has five boundaries, and so on. Let k be the total number of boundaries. Then

$$2k = 4c_4 + 5c_5 + 6c_6 + \ldots, \tag{1}$$

since each boundary has been counted twice. Moreover, if e denotes the total number of vertices, then

$$3e = 4c_4 + 5c_5 + 6c_6 + \ldots, \tag{2}$$

since exactly three countries meet at each vertex. Comparing (1) and (2), we find that

$$3e = 2k.$$

But according to a basic combinatorial theorem of Euler,

$$e + f = k + 2 \tag{3}$$

where f is the number of "faces," in our case "countries" (verify

this formula for a cube). Since the number of countries is just $2c$, (3) becomes

$$e + 2c = k + 2. \tag{4}$$

Multiplying both sides of (4) by 6, we obtain

$$6e + 12c = 6k + 12,$$

i.e.,

$$12c = 3e + 12$$

or

$$12(c_4 + c_5 + c_6 + \cdots) = (4c_4 + 5c_5 + 6c_6 + \cdots) + 12,$$

which implies

$$8c_4 + 7c_5 + \cdots + c_{11} = 12 + c_{13} + 2c_{14} + \cdots.$$

It follows (why?) that there exists at least one country a with no more than 11 vertices. Near of the two parts making up country a, there is a country sharing exactly one boundary with the given part. Assign each part of country a to the country sharing a boundary with this part. This gives a new map containing one less country. If the new map can be colored with 12 colors, so can the old map. Continuing this argument, we eventually arrive at a map containing a total of 12 countries. This map can obviously be colored with 12 colors, and hence so can the original map. Figure 56 shows an example of a map

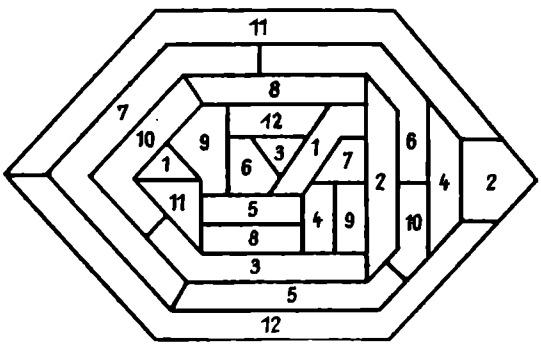

Fig. 56

for which 11 colors are not enough, since each of the 12 countries shares a boundary with the remaining 11.

143. Imagine the map of the planets drawn on two spherical globes, each made of a rubber sheet. By deforming the spheres, we convert them into hemispheres in such a way that the relative position of the countries is not changed but the same country lies along the equator and plane surface of each hemisphere (see Fig. 57). Imagine now that the hemispheres are joined into

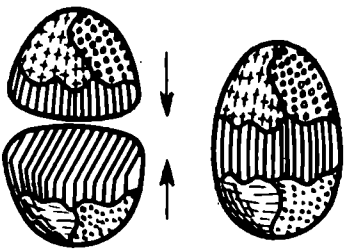

Fig. 57

a single sphere, we arrive at the same situation as in the preceding problem, where it was shown that a spherical map can be colored with 12 colors. Therefore the original map can also be colored with 12 colors. 7 colors are not enough, as shown by Figure 58, where 8 countries occupy two planets and each country shares a boundary with the remaining 7.

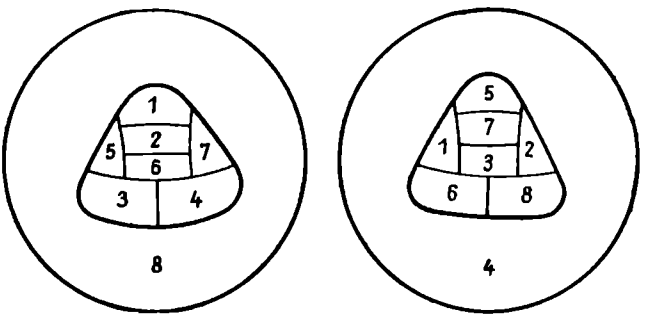

Fig. 58

Supplementary Problems

1. Find a six-digit number whose product with each of the numbers 2, 3, 4, 5 and 6 consists of the same digits written in a different order.

2. Can a right triangle with integral sides be drawn on a piece of graph paper in such a way that its vertices lie at nodes (points of intersection of the rulings) but none of its sides coincide with rulings? (Let the distance between consecutive rulings equal 1.)

3. Find all positive integers N such that the initial digits of the number N^2 coincide with N.

4. Choose an arbitrary point M on the segment AB, and draw the squares $AMDE$ and $MBCH$ lying on the same side of AB. Find the locus of the midpoints of the segment OO_1 (as M varies), where O and O_1 are the centers of the squares (see Fig. 59).

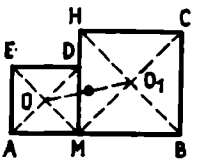

Fig. 59

5. Find the sum of the coefficients of the polynomial obtained by multiplying out the expression

$$(1 - 3x + 3x^2)^{743}(1 + 3x - 3x^2)^{744}$$

and collecting terms involving the same powers of x.

6. Prove that there is no order in which the integers from 1 to 1964 can be written such that the resulting number is a perfect cube.

7. Consider a quadrilateral inscribed in a circle, with one diagonal a diameter of the circle. Prove that the projections of the opposite sides of the quadrilateral onto the other diagonal are equal.

8. Fifteen newspapers lie on a table and completely cover it. Prove that 7 newspapers can be removed, with at least $\frac{8}{15}$ of the area of the table left covered by the remaining newspapers.

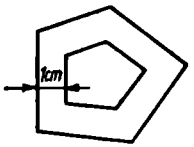

Fig. 60

9. The sides of a convex polygon of perimeter p are all "moved back" a distance of 1 cm. in the way shown in Figure 60. Prove that this increases the area of the polygon by more than $p + \pi$.

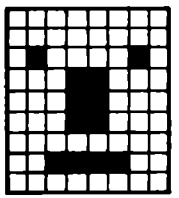

Fig. 61

10. Are there more numbers from 1 to 1,000,000 with 1 as a digit or more without 1 as a digit?

11. Prove that the figure shown in Figure 61 consisting of 60 squares cannot be broken up into rectangles consisting of three squares.

12. Find all relatively prime integers a and b such that

$$\frac{a+b}{a^2-ab+b^2}=\frac{3}{13}.$$

13. Let p_n denote the nth prime number. Prove that $p_n > 3n$ if $n \geqslant 12$.

14. A circle is divided into 256 sectors. Prove that eight-digit numbers consisting only of ones and twos (e.g., 12212111, 22121121) can be written in each sector in such a way that

1) No two numbers are the same;
2) Every two numbers appearing in adjacent sectors differ in only one digit.

15. Given a triangle ABC, draw the altitudes AK and BH, and let O be the center of the circumscribed circle. Prove that OC is perpendicular to KH.

16. Inside an arbitrary trapezoid find the point M for which the sum of the distances from M to the sides of the trapezoid is the smallest (if there are several such points, find them all).

17. Choose a point P on the bisector of a right angle, and through P draw any line cutting off segments of length a and

b from the sides of the angle. Prove that the quantity

$$\frac{1}{a} + \frac{1}{b}$$

is independent of the line drawn through P.

18. Prove that the area of a quadrilateral with sides a, b, c and d does not exceed

$$\frac{ac + bd}{2}.$$

19. A number which does not terminate in a zero is reduced an integral number of times as a result of deleting one of its digits. In what position can the deleted digit appear?

20. In a certain country there are direct railroad connections between any two cities, but only in one direction. Prove that there is a city which can be reached from any other city without going through more than one intermediate city.

21. Positive numbers are written in a rectangular table in such a way that the sum of the numbers in any row multiplied by the sum of the numbers in any column equals the number appearing at the intersection of the row and column. Prove that the sum of all the numbers in the table equals 1.

22. All the diagonals are drawn in a convex 17-gon, dividing it into subpolygons. What is the largest number of sides of any of these subpolygons?

23. Prove that the number $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

24. Given three points A, B and K, how should a line be drawn through K in order to make the sum of the distances from A and B to the line

1) As small as possible;

2) As large as possible?

25. Solve the equation

$$\sqrt{a - \sqrt{a + x}} = x.$$

26. Consider a game in which one of two players thinks of a

sequence of positive integers $a_1, a_2, \ldots, a_n$, and the other player tries to guess the sequence in the following way: He gives the first player another sequence $b_1, b_2, \ldots, b_n$ and then the first player tells him the sum

$$S_1 = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Then the second player gives the first another sequence $c_1, c_2, \ldots, c_n$ and the first player tells him the sum

$$S_2 = a_1c_1 + a_2c_2 + \cdots + a_nc_n.$$

What sequences should the second player choose in order to determine the sequence $a_1, a_2, \ldots, a_n$ correctly in just two tries?

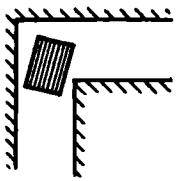

Fig. 62

27. A rectangular raft drifts along a canal of width h, making the 90° turn shown in Figure 62. What is the largest area the raft can have?

28. In the Martian language, the alphabet consists of just two letters P and S, and every two words of the same length differ in at least three letters. Prove that there are no more than 16 words of length 7.

29. Given a convex hexagon, suppose each of the three diagonals joining opposite vertices divides the area of the hexagon in half. Prove that the three diagonals intersect in a single point.

30. A closed ski course is constructed in such a way that it intersects itself several times without ever traversing the same stretch twice or passing through the same point three times. Before the meet begins, the judges go along the course and insert stakes numbered 1, 2, ... in order at the points of self-intersection. Prove that an odd and an even number appear at each intersection.

31. Some numbers whose sum is positive are arranged along the circumference of a circle. Prove that there is a number which is itself positive, whose sum with the next number in the clockwise direction is positive, whose sum with the next two numbers is positive, and so on.

32. Prove that the polynomial

$$x^{9999} + x^{8888} + x^{7777} + x^{6666} + x^{5555} + x^{4444} + x^{3333} + x^{2222} + x^{1111}$$

is divisible by

$$x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

33. Using a compass, construct the four vertices of the square inscribed in a given circle.

34. Find the four-digit number which leaves remainders 112 and 98 when divided by 131 and 132.

35. A number of circles of various diameters are drawn inside a square. Prove that the square can always be cut up into convex polygons such that each contains exactly one circle.